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Bayesian Estimations for the Two–parameter Exponential Model under the Type–II Censoring

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ABSTRACT

Suppose that we have two populations(or systems), say π_1 and π_2, to be tested. A random sample of size n from each population is taken and the test for each system will be terminated when the first r failures among n random samples are observed. This kind of test is called the type–II censored (or item–censored) testing without replacement. Under this scheme we consider the problem of estimating the unknown parameters of interests and the reliability for a given time t for each population.

1. Introduction

The exponential distribution is one of the most frequently used distributions and takes a very important role in the reliability (refer to Barlow and Proschan(1965)). Hence there is a huge body of literatures concerned with exponential models in the life–time and reliability analysis. For example, the cases that the failure time X follows one or two–parameter exponential distribution for a

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single population were discussed by Sinha and Guttman (1976) under the Bayesian setting. The classical statistical estimation of the reliability function was considered by Kurkjian and Karson(1987). Also for the type–II censored case of the one–parameter exponential distribution, Kambo(1978) studied the classical estimation of the reliability function. For the type–II doubly censored samples from the one–parameter exponential distribution, Shetty and Joshi(1987) derived classical estimators of the reliability function.

In this paper, we consider the case that the failure time X^i for the population π_i follows a two–parameter exponential distribution with a unknown location parameter μ_i and common unknown scale parameter θ, $i = 1, 2$, respectively. The problem of estimating parameters of interests including the reliability is formulated under the Bayesian approach. Also we consider the problem of selecting better one between two populations in terms of mean life–time.

In Section 2, we propose some Bayes estimators for μ_1, μ_2, and θ and the reliabilities for each population.

In Section 3, we consider the procedure selecting better one between two populations in terms of the mean life time under the Bayesian setting.

2. Some Proposed Bayes Estimators

We consider the two–parameter exponential distributions $\mathcal{E}(\mu_i, \theta)$, with location parameter μ_i, $i = 1, 2$, respectively, and a common scale parameter θ, whose probability density functions(pdf) are given by

$$f(x^i|\mu_i, \theta) = \frac{1}{\theta} \exp\left(-\frac{x^i - \mu_i}{\theta}\right), \quad i = 1, 2, \tag{1}$$

where $0 < \mu_i \leq x^i$, $0 < \theta < \infty$. Note that the μ_i is called the mean life time of the system i. A random sample of size n from each system is subjected to test and the test for each system is terminated when the first $r(\leq n)$ items fail. Let $X_1^i, X_2^i, \cdots, X_r^i$ be the failure times and let $X_{(1)}^i \leq X_{(2)}^i \cdots \leq X_{(r)}^i$ be ordered failure times from the system i, $i = 1, 2$, respectively. Then the likelihood function of μ_1, μ_2, and θ given $\underline{x} = \{X^i_{(1)} = x^i_{(1)}, X^i_{(2)} = x^i_{(2)}, \cdots, X^i_{(r)} = x^i_{(r)}, i = 1, 2\}$ is

$$L(\mu_1, \mu_2, \theta|\underline{x}) = \prod_{i=1}^{2}\left[\left(\exp\left(-\frac{x^i_{(r)} - \mu^i}{\theta}\right)\right)^{n-r} \prod_{j=1}^{r}\frac{1}{\theta}\exp\left(-\frac{x^i_{(j)} - \mu^i}{\theta}\right)\right] \tag{2}$$
$$= \frac{1}{\theta^{2r}}\exp\left[-\frac{1}{\theta}\left(S_r + n(x^1_{(1)} - \mu_1) + n(x^2_{(1)} - \mu_2)\right)\right],$$

where

$$S_r = \sum_{i=1}^{2}\sum_{j=1}^{r}(x^i_{(j)} - x^i_{(1)}) + (n-r)\sum_{i=1}^{2}(x^i_{(r)} - x^i_{(1)}).$$

Note that $x^1_{(1)}, x^2_{(1)}$, and $S_r/2r$ are maximum likelihood estimators(M.L.E.) of μ_1, μ_2, and θ, respectively.

For the prior distribution, the Jeffrey–type noninfomative prior (Martz and waller 1982), which is given by

$$\pi(\mu_1, \mu_2, \theta) \propto \frac{1}{\theta^a}, \qquad a \geq 0,$$

is considered. Then the posterior distribution of μ_1, μ_2, and θ given $\underline{x} = \{X^i_{(1)} = x^i_{(1)}, X^i_{(2)} = x^i_{(2)}, \cdots, X^i_{(r)} = x^i_{(r)}, i = 1, 2\}$ is

$$\pi(\mu_1, \mu_2, \theta|\underline{x}) = \frac{K}{\theta^{2r+a}}\exp\left[-\frac{1}{\theta}\left(S_r + n(x^1_{(1)} - \mu_1) + n(x^2_{(1)} - \mu_2)\right)\right], \tag{3}$$

where

$$K = \frac{n^2}{\Gamma(2r+a-3)}S_r^{2r+a-3}C_{2r}$$

and

$$C_{2r} = \left[1 - \left(1 + \frac{nx^1_{(1)}}{S_r}\right)^{-2r-a+3} - \left(1 + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3}\right.$$
$$\left. + \left(1 + \frac{nx^1_{(1)}}{S_r} + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3}\right]^{-1}.$$

Here $x^1_{(1)}, x^2_{(1)}$, and $S_r/(2r+a)$ are generalized maximum likelihood estimators(G.M.L.E.) of μ_1, μ_2, and θ, respectively. Note that the G.M.L.E. is the Bayesian version of the M.L.E. with the posterior density.

Remark. One can see that the G.M.L.E.'s are the same as the M.L.E.'s when $a = 0$.

Also the marginal posterior densities of μ_1, μ_2, and θ can be obtained easily and are given by

$$\begin{aligned}\pi(\mu_i|\underline{x}) &= n(2r+a-3)S_r^{-1}C_{2r} \\ &\quad\times\left[\left(1+\frac{nx^i_{(1)}}{S_r}-\frac{n\mu_i}{S_r}\right)^{-2r-a+2} - \left(1+\frac{nx^i_{(1)}}{S_r}+\frac{nx^j_{(1)}}{S_r}-\frac{n\mu_i}{S_r}\right)^{-2r-a+2}\right], \\ &\quad 0<\mu_i<x^i_{(1)},\quad i,j=1,2,\quad i\neq j,\end{aligned} \tag{4}$$

and

$$\begin{aligned}\pi(\theta|\underline{x}) &= \frac{1}{\theta^{2r+a-2}}\frac{1}{\Gamma(2r+a-3)}C_{2r}S_r^{2r+a-3} \\ &\quad\times\exp\left(-\frac{S_r}{\theta}\right)\left[1-\exp\left(-\frac{nx^1_{(1)}}{\theta}\right)\right]\left[1-\exp\left(-\frac{nx^2_{(1)}}{\theta}\right)\right], \\ &\quad 0<\theta<\infty.\end{aligned} \tag{5}$$

Hence with the squared error loss function, the following theorem can be obtained easily.

Theorem 1. Suppose that $\underline{x} = \{X^i_{(1)} = x^i_{(1)}, X^i_{(2)} = x^i_{(2)}, \cdots, X^i_{(r)} = x^i_{(r)}, i = 1, 2\}$ is a type–II censored sample from $\mathcal{E}(\mu_i, \theta)$, $i = 1, 2$, respectively. Let the joint prior of μ_1, μ_2, and θ be $\pi(\mu_1, \mu_2, \theta) \propto 1/\theta^a$, $a \geq 0$. Then under the squared error loss the Bayes estimators $\hat{\mu}_i$ are

$$\begin{aligned}\hat{\mu}_i &= \frac{C_{2r}}{n(2r+a-4)} \\ &\quad\times\left\{n(2r+a-4)x^i_{(1)}\left[1-\left(1+\frac{nx^j_{(1)}}{S_r}\right)^{-2r-a+3}\right]-S_rC^{-1}_{2r-1}\right\}, \\ &\quad i,j=1,2,\quad i\neq j,\end{aligned}$$

and the Bayes estimator $\hat{\theta}$ of θ is

$$\hat{\theta} = \frac{C_{2r}S_r}{2r+a-4}C_{2r-1}^{-1}, \qquad 2r+a>4.$$

Proof. Under the squared error loss, the Bayes estimator is the posterior means. Thus, by using the inverted gamma function, we can obtain the estimators.

Now consider a Bayes estimator of the reliability function of the system i at a given time t, which is

$$R_t^i = P(X^i > t) = \exp\left(-\frac{t-\mu_i}{\theta}\right), \quad t \geq \mu_i,$$

$i = 1, 2$, respectively. The Bayes estimators $\widehat{R}_t^i$ of R_t^i under the squared error loss can be obtained and are given in the following theorem.

Theorem 2. Under the assumptions of Theorem 1, the joint posterior density of μ_i and θ, is

$$\pi(\mu_i, \theta|\underline{x}) = \frac{K}{n\theta^{2r+a-1}} \exp\left[-\frac{1}{\theta}(S_r + n(x_{(1)}^i - \mu_i))\right] \times \left[1 - \exp\left(-\frac{nx_{(1)}^j}{\theta}\right)\right], \quad i,j = 1,2, \quad i \neq j \tag{6}$$

and the Bayes estimator of the reliability function $R_t^i = P(X^i > t)$ is

$$\begin{aligned}\widehat{R}_t^i = \frac{n}{n+1}C_{2r}\Bigg[&\left(1+\frac{t}{S_r}-\frac{x_{(1)}^i}{S_r}\right)^{-2r-a+3} \\ &-\left(1+\frac{t}{S_r}+\frac{nx_{(1)}^i}{S_r}\right)^{-2r-a+3} \\ &-\left(1+\frac{nx_{(1)}^j}{S_r}+\frac{t}{S_r}-\frac{x_{(1)}^i}{S_r}\right)^{-2r-a+3} \\ &+\left(1+\frac{nx_{(1)}^i}{S_r}+\frac{nx_{(1)}^j}{S_r}+\frac{t}{S_r}\right)^{-2r-a+3}\Bigg], \quad i,j = 1,2, \quad i \neq j.\end{aligned}$$

Proof. The joint posterior density function of μ_i, and θ, $i=1,2$, is

$$\pi(\mu_i,\theta|\underline{x}) = \int_0^{x^j_{(1)}} \pi(\mu_1,\mu_2,\theta|\underline{x})\,d\mu_j, \quad i,j=1,2, \quad i\neq j.$$

Thus $\pi(\mu_i,\theta|\underline{x})$ can be obtained by usual computations. Therefore under the squared error loss, the Bayes estimators of the reliability functions at given time t, $R^i_t = P(X^i > t)$, $i=1,2$, are the posterior mean of $R^i_t = \exp(-(t-\mu_i)/\theta)$, $i=1,2$, given $\underline{x}$.

Also one can obtain the posterior distribution of R^i_t, $i=1,2$ by using the following transformation $R^i_t = \exp(-(t-\mu_i)/\theta)$ and $\nu=\theta$.

Theorem 3. Under the assumptions of Theorem 1, the posterior density of $R^i_t = \exp(-(t-\mu_i)/\theta)$ is given by

$$\begin{aligned}\pi(R^i_t|\underline{x}) = &\frac{K}{n}(R^i_t)^{n-1}\Gamma(2r+a-3) \\ &\times\Bigg[\frac{1}{W_{r1}^{2r+a-3}}\exp\left(-\frac{W_{r1}}{\alpha_2}\right)\sum_{m=0}^{2r+a-4}\left(\frac{W_{r1}}{\alpha_2}\right)^m/m! \\ &-\frac{1}{W_{r1}^{2r+a-3}}\exp\left(-\frac{W_{r1}}{\alpha_1}\right)\sum_{m=0}^{2r+a-4}\left(\frac{W_{r1}}{\alpha_1}\right)^m/m! \\ &-\frac{1}{W_{r2}^{2r+a-3}}\exp\left(-\frac{W_{r2}}{\alpha_2}\right)\sum_{m=0}^{2r+a-4}\left(\frac{W_{r2}}{\alpha_2}\right)^m/m! \\ &+\frac{1}{W_{r2}^{2r+a-3}}\exp\left(-\frac{W_{r2}}{\alpha_1}\right)\sum_{m=0}^{2r+a-4}\left(\frac{W_{r2}}{\alpha_1}\right)^m/m!\Bigg], \quad i=1,2,\end{aligned} \tag{7}$$

where $\alpha_1 = \dfrac{t-x^i_{(1)}}{\ln(R^i_t)^{-1}}$, $\alpha_2 = \dfrac{t}{\ln(R^i_t)^{-1}}$,

$W_{r1} = S_r + nx^1_{(1)} - nt$, $W_{r2} = S_r + nx^1_{(1)} + nx^2_{(1)} - nt$.

Proof. From Theorem 2, the joint posterior density of μ_i and θ, $i=1,2$, is

$$\begin{aligned}\pi(\mu_i,\theta|\underline{x}) = &\frac{K}{n\theta^{2r+a-1}}\exp\left[-\frac{1}{\theta}\left(S_r + n(x^i_{(1)}-\mu_i)\right)\right] \\ &\times\left[1-\exp\left(-\frac{n}{\theta}x^j_{(1)}\right)\right],\end{aligned}$$

for $0 < \mu_i < x^i_{(1)}, \quad 0 < \theta < \infty, \quad i,j = 1,2, \quad i \neq j$. Then the transformation g is one-to-one on $S = \{(\mu_i, \theta) : 0 < \mu_i < x^i_{(1)}, \theta > 0\}$ and its range is

$$S_1 = \left\{(R^i_t, \nu) : \exp\left(-\frac{t}{\nu}\right) < R^i_t < \exp\left(-\frac{t - x^i_{(1)}}{\nu}\right), \ \frac{t - x^i_{(1)}}{\ln(R^i_t)^{-1}} < \nu < \frac{t}{\ln(R^i_t)^{-1}}\right\}.$$

Note that on S_1, $g^{-1}(R^i_t, \nu) = (t + \nu \ln R^i_t, \nu)$.

Thereforem the jacobian of the transformation g is

$$J_{g^{-1}}(R^i_t, \nu) \begin{vmatrix} \nu / R^i_t & \ln R^i_t \\ 0 & 1 \end{vmatrix} = \frac{\nu}{R^i_t}.$$

Then the joint posterior density of R^i_t and ν is given by

$$\pi(R^i_t, \nu | \underline{x}) = \pi(t + \nu \ln R^i_t, \nu) |J|,$$

for $\exp(-\frac{t}{\nu}) < R^i_t < \exp\left(\frac{t - x^i_{(1)}}{\nu}\right)$, and $\frac{t - x^i_{(1)}}{\ln(R^i_t)^{-1}} < \nu < \frac{t}{\ln(R^i_t)^{-1}}$. Using the identity connecting the incomplete gama distribution with Poisson, (7) is obtained.

Remark. One can get the Bayes estimator of R^i_t from the posterior density in the equation (7) of Theorem 4 directly.

3. The Proposed Bayes Procedure Selecting Better One

In this section, we consider the problem of selecting better one between two populations in terms of the mean life time under the Bayesian setting.

With the $0 - K_i$ loss (Berger 1985), the Bayes rule for selecting better one is as follows

$\mathcal{R}$: Select the i–th system if

$$\frac{K_0}{K_1} > \frac{P(\mu_j > \mu_i | \underline{x})}{P(\mu_i \geq \mu_j | \underline{x})}, \quad i,j = 1,2, \quad i \neq j,$$

or equivalently

select the i–th system if

$$P(\mu_i \geq \mu_j | \underset{\sim}{x}) > \frac{K_1}{K_0 + K_1}, \quad i \neq j.$$

For $K_0 = K_1 = 1$, *i.e.*, the 0–1 loss, the Bayes rule for selecting better one is as follows :

$\mathcal{R}$: Select the i–th system if

$$P(\mu_i \geq \mu_j | \underset{\sim}{x}) > P(\mu_j \geq \mu_i | \underset{\sim}{x}), \quad i, j = 1, 2, \quad i \neq j.$$

or equivalently

select the i–th system if

$$P(\mu_i \geq \mu_j | \underset{\sim}{x}) > 1/2, \quad i \neq j.$$

If $P(\mu_i \geq \mu_j | \underset{\sim}{x}) = 1/2$, select one population randomly.

Now to evaluate the quantity $P(\mu_i \geq \mu_j | \underset{\sim}{x})$, the joint posterior density of μ_1 and μ_2 given $\underset{\sim}{x} = \{X^i_{(1)} = x^i_{(1)}, X^i_{(2)} = x^i_{(2)}, \cdots, X^i_{(r)} = x^i_{(r)}, i = 1, 2\}$ is derived from the equation (3) and is given by

$$\begin{aligned} \pi(\mu_1, \mu_2 | \underset{\sim}{x}) &= n^2(2r + a - 2)(2r + a - 3) S_r^{-2} C_{2r} \\ &\times \left(1 + \frac{n(x^1_{(1)} - \mu_1)}{S_r} + \frac{n(x^2_{(1)} - \mu_2)}{S_r}\right)^{-2r-a+1} \\ &0 < \mu_1 < x^1_{(1)}, \quad 0 < \mu_2 < x^2_{(1)}. \end{aligned} \tag{8}$$

Here two cases, *i.e.*, (1) $x^1_{(1)} < x^2_{(1)}$ and (2) $x^1_{(1)} > x^2_{(1)}$ are considered.

For each case $P(\mu_1 \geq \mu_2 | \underset{\sim}{x})$ is computed and is given in the following theorem.

Theorem 4. Under the assumptions of Theorem 1, the joint posterior density of μ_1 and μ_2 is

$$\pi(\mu_1, \mu_2|\underline{x}) = n^2(2r+a-2)(2r+a-3)S_r^{-2}C_{2r} \times \left(1 + \frac{n(x^1_{(1)} - \mu_1)}{S_r} + \frac{n(x^2_{(1)} - \mu_2)}{S_r}\right)^{-2r-a+1}, \quad 0 < \mu_1 < x^1_{(1)}, \ 0 < \mu_2 < x^2_{(1)}, \tag{9}$$

and for $x^1_{(1)} < x^2_{(1)}$,

$$P(\mu_1 \geq \mu_2|\underline{x}) = \frac{C_{2r}}{2}\left[\left(1 + \frac{nx^2_{(1)}}{S_r} - \frac{nx^1_{(1)}}{S_r}\right)^{-2r-a+3} + \left(1 + \frac{nx^1_{(1)}}{S_r} + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3} - 2\left(1 + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3}\right],$$

and for $x^1_{(1)} > x^2_{(1)}$,

$$P(\mu_1 \geq \mu_2|\underline{x}) = \frac{C_{2r}}{2}\left[\left(1 + \frac{nx^1_{(1)}}{S_r} + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3} - \left(1 + \frac{nx^1_{(1)}}{S_r} - \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3} + 2 - 2\left(1 + \frac{nx^2_{(1)}}{S_r}\right)^{-2r-a+3}\right].$$

References

1. Barlow, R.E. and Proschan, F.(1965), *Mathematical Theory of Reliability*, John Wiley & Sons, New York.

2. Berger, J.O.(1985), *Statistical Decision Theory and Bayesian Analysis*, Springer–Verlag, New York.

3. Kambo, N.S.(1978), Maximum Likelihood Estimators of the Location and Scale Parameters of the Exponential Distribution from a Censored Sample, *Commun. Statist.-Theor. Meth.*, A5(5), 471–479.

4. Kurkjian, B.M. and Karson, M.J.(1987), Reliability Estimation for the Exponential Distribution, *Commun. Statist.-Theor. Meth.*, 16(3), 835–853.

5. Martz, H.F. and Waller, R.A.(1982), *Bayesian Reliability Analysis*, John Wiley & Sons, New York.

6. Shetty, B.N. and Joshi, P.C.(1987), Estimation of Parameters of k Exponential Distributions in Doubly Censored Samples, *Commun. Statist.-Theor. Meth.*, 16(7), 2115–2123.

7. Sinha, S.K. and Guttman, I.(1976), Bayesian Inference about the Reliability Function for the Exponential Distributions, *Commun. Statist.-Theor. Meth.*, A5(5), 471–479.