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Bayesian Reliability Estimation for the Rayleigh Distribution

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ABSTRACT

This paper deals with the problem of estimating a reliability function for the Rayleigh distribution. Using the priors about a reliabity of real interest some Bayes estimators and Bayes credible sets are proposed and studied under squared error loss and Harris loss.

1. Introduction

In the context of the lifetime of industrial equipments and components, the problem of estimating a reliability function plays an important role in many practical reliability analysis. In this paper we deal with the Bayesian point estimation and the Bayesian interval estimation for the reliability function, at a specified mission time t, based upon a complete sample of failure times observed from the Rayleigh model. Siddiqui(1962) discussed the origin and the properties of the Rayleigh distribution. Polovko(1968) noted the importance of this distribution in electrovaccum devices. Dyer and Whisenand(1973a, 1973b) mentioned the important role of the Rayleigh distribution in the area of the communication engineering. Cheng(1980) investigated the optimum spacing for asymptotically best linear unbiased estimator of the parameter based upon order statistics.

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Although the classical statistical estimation procedures have been applied for many situations, there are many cases in which the Bayesian procedures perform more satisfactory. The benefits of Bayesian approach in the reliability analysis were discussed by Evans(1969).

The purpose of this paper is to propose and to study some Bayes estimators and Bayesian credible regions of a reliability function with respect to the direct prior information about the reliability for the Rayleigh model. As prior distributions, we consider a locally uniform prior distribution and a beta prior distribution. The squared error loss function is used typically, but it is sometimes appropriate in analyzing reliability data. Thus the loss function suggested by Harris(1976) as well as the squared error loss function are considered.

In Section 2, we derive some Bayes estimators of the reliability function under the squared error loss function and Harris loss function. In Section 3, we propose the Bayesian credible intervals and the highest posterior density (HPD) credible regions for the reliability function with respect to a locally uniform prior and a beta prior.

2. Some Proposed Bayes Estimators

We consider the Bayesian approach of the estimation of reliability, at a specified time t, for the Rayleigh distribution, denoted by $\mathcal{R}(\sigma^2)$, with probability density function(pdf) given in (2.1).

$$f(x|\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad 0 < x < \infty, \tag{2.1}$$

This is the special case of the two-parameter Weibull distribution. The hazard function of this distribution is an increasing function in x, which is interested in the life testing problem. Thus this could be suitable for life testing experiments on components which age with time in that way.

For the Rayleigh distribution, the reliability function θ at a specified 'mission' time $t > 0$ is given by

$$\theta = P(X > t) = \exp\left(\frac{-t^2}{2\sigma^2}\right). \tag{2.2}$$

Then our goal is to derive a Bayes estimator of the reliability function θ.

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample from the Rayleigh distribution with pdf given in (2.1). Then the likelihood function is given by

$$L(\sigma|\underline{X}) = \frac{1}{\sigma^{2n}} \prod_{i=1}^{n} X_i \exp\left(-\frac{\sum X_i^2}{2\sigma^2}\right), \quad 0 < \sigma, \ 0 < X_i < \infty, \ i = 1, 2, \cdots, n.$$

Thus, based upon a random sample $\underline{X}$ from $\mathcal{R}(\sigma^2)$, the likelihood function of θ , which can be obtained from (2.2) by letting $\sigma^2 = -t^2/2\ln\theta$, is

$$\begin{aligned} L(\theta|\underline{X}) &= \left(\frac{-2\ln\theta}{t^2}\right)^n \prod_{i=1}^{n} X_i \exp\left(\frac{\sum X_i^2}{t^2}\ln\theta\right) \\ &\propto (-\ln\theta)^n \exp\left(\frac{\sum X_i^2}{t^2}\ln\theta\right), \ 0 < \theta < 1. \end{aligned} \tag{2.3}$$

Here we consider the Jeffreys(1961)' noninformative prior for θ. When the mission time t is given, the noninformative prior $\pi(\cdot)$ for θ is

$$\Pi(\theta) \propto I(\theta)^{\frac{1}{2}} = -\frac{1}{\theta\ln\theta},$$

where

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln L(\theta|\underline{X})\right] = \frac{1}{\theta^2(\ln\theta)^2}, \quad 0 < \theta < 1,$$

and $I(\theta)$ is the Fisher's information. This is also the implied prior of the noninformative prior for σ, which is given by Sinha and Howlader(1983), by the properties of the invariance under parametric transformations. Therefore the Bayes estimators under the noninformative prior for θ and σ are equivalent. Sinha and Howlader(1983) proposed the Bayes estimator $\hat{\theta}_{NS}$ of θ under squared error loss, which is given by

$$\hat{\theta}_{NS} = \left(1 + \frac{t^2}{\sum X_i^2}\right)^{-n}.$$

Note that the Bayes estimator with respect to a noninformative prior is approximately equal to the maximum likelihood estimator $\exp\left(-\frac{nt^2}{\sum X_i^2}\right)$ for large sample size.

Next, we consider a loss function suggested by Harris(1976) for the case $k = 2$, given by

$$L(\theta, \delta) = \left| \frac{1}{1-\delta} - \frac{1}{1-\theta} \right|^2. \tag{2.4}$$

Under the Harris loss, the Bayes estimator $\hat{\theta}_H$ of θ can be drived from the followings: for a given $\underline{X} = \underline{x}$,

$$\frac{1}{1-\delta} = E^{\theta|\underline{x}}\left[\frac{1}{1-\theta}\right] \equiv \gamma(\underline{x}).$$

Thus

$$\hat{\theta}_H = 1 - \frac{1}{\gamma(\underline{x})}, \tag{2.5}$$

where

$$\gamma(\underline{x}) = E^{\theta|\underline{x}}\left[\frac{1}{1-\theta}\right].$$

Then one can obtain the Bayes estimator of the reliability function as follows:

Theorem 2.1. Under the Harris loss, the Bayes estimator $\hat{\theta}_{NH}$ of the reliability function θ is given by

$$\hat{\theta}_{NH} = 1 - \frac{1}{\left(\frac{\sum X_i^2}{t^2}\right)^n \sum_{m=0}^{\infty}\left(m + \frac{\sum X_i^2}{t^2}\right)^{-n}}.$$

Proof. The posterior density of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = K(-\ln\theta)^{n-1}\theta^{\frac{\sum x_i^2}{t^2}-1}, \quad 0 < \theta < 1,$$

where K is the normalizing constant and is given by $\frac{1}{\Gamma(n)}\left(\frac{\sum x_i^2}{t^2}\right)^n$, and $\Gamma(a)$ is the gamma function. Then we have

$$E^{\theta|\underline{x}}\left(\frac{1}{1-\theta}\right) = K\int_0^1 \frac{1}{1-\theta}\theta^{\frac{\sum x_i^2}{t^2}-1}(-\ln\theta)^{n-1}\,d\theta.$$

Letting $Y = -\ln\theta\left(1+\frac{\sum X_i^2}{t^2}\right)$,

$$E^{\theta|\underline{x}}\left(\frac{1}{1-\theta}\right) = \frac{1}{\Gamma(n)\left(1+\frac{t^2}{\sum x_i^2}\right)^n}\int_0^\infty y^{n-1}\frac{\exp\left(\frac{-y}{1+t^2/\sum x_i^2}\right)}{1-\exp\left(\frac{-y}{1+\sum x_i^2/t^2}\right)}\,dy. \quad (2.6)$$

With the aid of the formula 4.5(10) in Erdélyi *et al.*(1955),

$$\int_0^\infty z^{\nu-1}(1-e^{-z/a})^{-1}e^{-pz}\,dz = a^\nu\Gamma(\nu)\zeta(\nu,ap) \quad \text{for } Re(p)>0,\ \ Re(\nu)>1,$$

where $\zeta(s,\nu) = \sum_{m=0}^\infty(\nu+m)^{-s}(Re(s)>0)$ is the generalized zeta function, the equation (2.6) becomes

$$\left(\frac{\sum x_i^2}{t^2}\right)^n\zeta\left(n,\frac{\sum x_i^2}{t^2}\right) = \left(\frac{\sum x_i^2}{t^2}\right)^n\sum_{m=0}^\infty\left(m+\frac{\sum x_i^2}{t^2}\right)^{-n}.$$

Hence the theorem follows from the equation (2.5).

Consider a locally uniform prior distribution $\mathcal{U}^\theta(0,1)$ for θ with pdf

$$\Pi(\theta) = 1, \qquad 0<\theta<1. \quad (2.7)$$

Then the posterior density of θ, given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{\Gamma(n+1)}\left(1+\frac{\sum x_i^2}{t^2}\right)^{n+1}(-\ln\theta)^n\exp\left(\frac{\sum x_i^2}{t^2}\ln\theta\right), 0<\theta<1. \quad (2.8)$$

If the squared error loss function is applied, then the Bayes estimator of the reliability function is given as follows:

Theorem 2.2. If the squared error loss and a locally uniform prior are used, then the Bayes estimator $\tilde{\theta}_{US}$ of the reliability function is given by

$$\tilde{\theta}_{US} = \left(1+\frac{t^2}{\sum X_i^2}\right)^{n+1}\left(1+\frac{2t^2}{\sum X_i^2}\right)^{-(n+1)}.$$

Proof. Since the Bayes estimator of θ is the posterior mean with the squared error loss,

$$\begin{aligned}\tilde{\theta}_{US} &= E^{\theta|\underline{x}}(\theta)\\ &= \frac{1}{\Gamma(n+1)}\left(1+\frac{\sum x_i^2}{t^2}\right)^{n+1}\int_0^1 \theta^{\sum x_i^2/t^2+1}(-\ln\theta)^n d\theta\end{aligned}$$

Letting $Y = -\ln\theta\left(2+\frac{\sum X_i^2}{t^2}\right)$,

$$\begin{aligned}\tilde{\theta}_{US} &= \frac{1}{\Gamma(n+1)}\left(1+\frac{\sum x_i^2}{t^2}\right)^{n+1}\left(2+\frac{\sum x_i^2}{t^2}\right)^{-(n+1)}\int_0^\infty y^n e^{-y}dy\\ &= \left(1+\frac{t^2}{\sum x_i^2}\right)^{n+1}\left(1+\frac{2t^2}{\sum x_i^2}\right)^{-(n+1)}.\end{aligned}$$

Also with the Harris loss, one can obtain the following theorem.

Theorem 2.3. If the Harris loss and a locally uniform prior distribution are used, then the Bayes estimator $\tilde{\theta}_{UH}$ of the reliability function is given by

$$\hat{\theta}_{UH} = 1 - \frac{1}{\left(1+\frac{\sum X_i^2}{t^2}\right)^{n+1}\sum_{m=0}^{\infty}\left(1+m+\frac{\sum X_i^2}{t^2}\right)^{-(n+1)}}.$$

Proof. By transforming $Y = -\theta^{\frac{\sum x_i^2}{t^2}}$, one can obtain the following relation:

$$E^{\theta|\underline{x}}\left(\frac{1}{1-\theta}\right) = \frac{1}{\Gamma(n+1)}\left(1+\frac{t^2}{\sum x_i^2}\right)^{n+1}\int_0^\infty y^n \frac{\exp\left[-y\left(1+\frac{t^2}{\sum x_i^2}\right)\right]}{1-\exp\left(-\frac{t^2}{\sum x_i^2}y\right)}dy.$$

Using 4.5(10) in Erdélyi *et al.*(1955), the above expectation is equal to

$$\left(1+\frac{\sum x_i^2}{t^2}\right)^{n+1}\sum_{m=0}^{\infty}\left(1+m+\frac{\sum x_i^2}{t^2}\right)^{-(n+1)}$$

Therefore the theorem follows immediately from the equation (2.5).

Remark. The generalized maximum likelihood estimator of the reliability is the largest mode of the posterior density function of θ given $\underline{X} = \underline{x}$. If the prior density of θ does not depend upon θ, then the posterior density function of θ is proportional to the likelihood function of σ. If a uniform prior $\mathcal{U}^{\theta}(0,1)$ is assigned on θ, then the generalized MLE is also the classical MLE, $\exp\left(-\frac{nt^2}{\sum X_i^2}\right)$, $t > 0$.

Now we consider a beta prior distribution $\mathcal{B}^{\theta}(\alpha, \beta)$ with pdf

$$\Pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0 < \theta < 1, \ 0 < \alpha, \ \beta,$$

where $B(\alpha, \beta)$ is the beta function with parameters α and β defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Combining the likelihood function in (2.3) and a beta prior density function, the posterior density function of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{\mathcal{I}(\alpha,\beta)}(-\ln\theta)^n \exp\left(\frac{\sum x_i^2}{t^2}\ln\theta\right)\theta^{\alpha-1}(1-\theta)^{\beta-1}, \ 0 < \theta < 1, \quad (2.9)$$

where

$$\mathcal{I}(\alpha,\beta) = \int_0^1 (-\ln\theta)^n \exp\left(\frac{\sum x_i^2}{t^2}\ln\theta\right)\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta. \quad (2.10)$$

With the squared error loss, the Bayes estimator of the reliability is the posterior mean and is given as follows:

Theorem 2.4. If the squared error loss function is used and θ has a beta prior distribution with parameters α and β, then the Bayes estimator $\tilde{\theta}_{BS}$ of the reliability function given $\underline{X} = \underline{x}$ is given by

$$\tilde{\theta}_{BS} = \frac{\mathcal{I}(\alpha+1,\beta)}{\mathcal{I}I(\alpha,\beta)},$$

where $\mathcal{I}(a,b)$ is given by the equation (2.10).

Proof. This can be easily proved, so we omit the proof.

Also with the Harris loss function in (2.4) the following theorem can be easily obtained.

Theorem 2.5. If the Harris error loss function is used and θ follows a beta prior distribution with parameters α and β, then the Bayes estimator $\tilde{\theta}_{BH}$ of the reliability function given $\underline{X} = \underline{x}$ is given by

$$\tilde{\theta}_{BH} = 1 - \frac{\mathcal{I}(\alpha,\beta)}{\mathcal{I}(\alpha,\beta-1)},$$

where $I(a,b)$ is given by the equation (2.10).

3. Credible Intervals and HPD Credible Regions

In this section we derive the equal–tail credible intervals and the Highest Posterior Density(HPD) credible intervals for the reliability function. A subset C of the parameter space, with a degree of confidence $(1-\alpha)$ based upon a posterior probability density function $\Pi(\theta|\underline{x})$ is called a credible set, i.e. $\int_C \pi(\theta|\underline{x})d\theta \geq 1-\alpha$. If the credible set is an interval, then it is known as a *credible interval* (Edwards, Lindman, and Savage(1963)) or a *Bayesian confidence interval* (Lindley(1965)). An equal–tail $100(1-\alpha)$percent confidence interval $(\hat{\theta}_L, \hat{\theta}_U)$ for the reliability function θ is obtain by solving $\int_{-\infty}^{\hat{\theta}_L} \pi(\theta|\underline{x})d\theta = \int_{\hat{\theta}_U}^{\infty} \pi(\theta|\underline{x})d\theta$.

It is natural to want to find the shorter credible set. A subset C of the parameter space Θ satisfying $C = \{\theta \in \Theta \mid \Pi(\theta|\underline{x}) \geq K(\alpha)\}$, where $K(\alpha)$ is the largest constant such that $P(C|\underline{x}) \geq 1-\alpha$, is called the $100(1-\alpha)$ percent highest probability density credible set (Berger(1985)).

First consider the Rayleigh distribution with pdf in (2.1). In this case, the likelihood function is given in (2.3). With the noninformative prior distribution on the reliability θ, the equal–tail credible interval and the HPD credible interval were proposed by Sinha and Howlader (1983) and are given in the following: With a noninformative prior on θ, the $100(1-\alpha)\%$ credible interval $(\tilde{\theta}_{NL},\ \tilde{\theta}_{NU})$ for θ is given by

$$\left(\exp\left(\frac{-\chi^2_{(2n;\alpha/2)}}{\frac{2}{t^2}\sum X_i^2}\right),\ \exp\left(\frac{-\chi^2_{(2n;1-\alpha/2)}}{\frac{2}{t^2}\sum X_i^2}\right)\right).$$

With a noninformative prior, the $100(1-\alpha)\%$ HPD credible bounds $\tilde{\theta}_{NHL}$ and $\tilde{\theta}_{NHU}$ are the simultaneous solutions of

$$1-\alpha = P\left[\frac{2\sum X_i^2}{t^2}\ln\left(\frac{1}{\tilde{\theta}_{NHU}}\right) < \chi^2_{2n} < \frac{2\sum X_i^2}{t^2}\ln\left(\frac{1}{\tilde{\theta}_{NHL}}\right)\right],$$

$$\tilde{\theta}_{NHL}^{\frac{2}{t^2}\sum X_i^2-1}\left(\ln\tilde{\theta}_{NHL}\right)^{n-1} = \tilde{\theta}_{NHU}^{\frac{1}{t^2}\sum X_i^2-1}\left(\ln\tilde{\theta}_{NHU}\right)^{n-1}.$$

Now we consider a locally uniform prior distribution on θ to construct the credible interval for the reliability.

From the equations (2.3) and (2.7) the posterior density of θ is in (2.8). We transform θ to w, where

$$w = -2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\theta.$$

The Jacobian of transformation is given by

$$|J| = \frac{1}{2(1+\frac{\sum X_i^2}{t^2})}\exp\left\{\frac{-w}{2(1+\frac{\sum X_i^2}{t^2})}\right\},$$

so that the pdf of w given $\underline{X} = \underline{x}$ is given by

$$\Pi(w|\underline{x}) = \frac{1}{\Gamma(n+1)}\left(\frac{1}{2}\right)^{n+1}w^n e^{-w/2}, \qquad 0 < w < \infty,$$

where $\Gamma(a)$ is the gamma function. That is, the distribution of w is the chi–square distribution with 2(n+1) degrees of freedom. Therefore the $100(1-\alpha)\%$ equal–tail credible bounds $\tilde{\theta}_{CL}$ and $\tilde{\theta}_{CU}$ for θ in (2.2) are obtained by solving

$$-2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\tilde{\theta}_{CU} = \chi^2_{(2(n+1);1-\alpha/2)} \tag{3.1}$$

and

$$-2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\tilde{\theta}_{CL}=\chi^2_{(2(n+1);\alpha/2)},\tag{3.2}$$

where $\chi^2_{(\nu;\alpha)}$ is the $100(1-\alpha)th$ percentiles of a chi–square distribution with ν degrees of freedom.

Then we obtain the following theorems.

Theorem 3.1. With a locally uniform prior on θ, the $100(1-\alpha)\%$ equal–tail credible interval $(\tilde{\theta}_{CL},\ \tilde{\theta}_{CU})$ is

$$\left(\exp\left(\frac{-\chi^2_{(2(n+1);\alpha/2)}}{2(1+\frac{1}{t}\sum X_i^2)}\right),\ \exp\left(\frac{-\chi^2_{(2(n+1);1-\alpha/2)}}{2(1+\frac{1}{t}\sum X_i^2)}\right)\right).$$

Proof. The theorem is easily proved by solving the equations (3.1) and (3.2).

Theorem 3.2. If a locally uniform prior distribution for θ is used, the $100(1-\alpha)\%$ HPD credible bounds $\tilde{\theta}_{HL}$ and $\tilde{\theta}_{HU}$ of the reliability function θ in (2.2) are the simultaneous solutions of

$$P\left[-2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\tilde{\theta}_{HU}<\chi^2_{2(n+1)}<-2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\tilde{\theta}_{HL}\right]=1-\alpha,$$

$$(-\ln\tilde{\theta}_{HL})^n\exp\left(\frac{\sum X_i^2}{t^2}\ln\hat{\theta}_{HL}\right)=(-\ln\tilde{\theta}_{HU})^n\exp\left(\frac{\sum X_i^2}{t^2}\ln\hat{\theta}_{HU}\right).$$

Proof. From the posterior densities of θ and w given $\underline{X}=\underline{x}$, the theorem is easily proved.

We assume that the reliability function θ follows a beta prior distribution, $B^\theta(\alpha,\beta)$. Then the posterior density of θ given $\underline{X}=\underline{x}$ is given in (2.9). For this case the equal–tail credible interval and the HPD credible region can not be obtained in a closed form. Therefore in order to obtain the credible intervals, some numerical techniques should be used for numerical integrations. For example, the International Mathematical and Statistical Libraries(IMSL) subroutine DCADRE may be a useful computer routine for the numerical integrations.

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