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Estimation of Mean Residual Life Function for a Coherent System[1]

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Abstract

In this paper we propose a nonparametric estimator of the mean residual life function (MRLF) on a coherent system under the condition that the component lifetimes are censored by system lifetime. It is shown that the proposed estimator, considered as a function of age t, converges weakly to a Gaussian process on a fixed interval. A consistent estimator of asymptotic variance of the proposed estimator is also given.

1. Introduction

Consider a coherent system of m independent components labeled as $1, 2, \ldots, m$. Let R_ϕ denote the reliability of the system which has the structure function ϕ, and $R_j(t)$ denote the reliability of component j, $1, 2, \ldots, m$, respectively.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed ($i.i.d.$) failure times of n systems with common continuous distribution function $F(t) = 1 - R_\phi(t)$ with $F(0) = 0$ and a finite mean, and T_{1j}, $T_{2j}, \ldots, T_{nj}$ be $i.i.d.$ failure

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times of the component j in n systems with common continuous distribution function $F_j(t) = 1 - R_j(t)$ with $F_j(0)$ and a finite mean. Then for each coherent structure ϕ of m independent components, we represent the MRLF at age t, $e_\phi(t)$, of the system as the function of the system reliability as

$$\begin{aligned} e_\phi(t) &= E(X - t \mid X > t) \\ &= (R_\phi(t))^{-1} \int_t^\infty R_\phi(u)\, du, \end{aligned}$$

where

$$R_\phi(t) = h_\phi(R_1(t), R_2(t), \ldots, R_m(t)).$$

Here, h_ϕ is called a reliability function of a system (Barlow and Proschan, 1981).

We also set an alternative of the MRLF, $e_T(t)$ for a coherent system using the system relibility $R_\phi(t)$ on a fixed interval on $[0, T]$, $T < \infty$ as follows;

$$e_T(t) = (R_\phi(t))^{-1} \int_t^T R_\phi(u)\, du.$$

The MRLF for a system playes a very important role in many practical engineering areas and in other applications. Hence we will consider estimation of $e_T(t)$ under a given censoring scheme.

For a system with one component, the estimation problem of the MRLF has been investigated by many researchers through parametric and nonparametric approaches since Cox (1961) and Swartz (1973) represented the MRLF at age t, $e(t)$ in terms of $E(X - t \mid X > t)$. In the case of no censoring, Yang (1978) investigated asymptotic properties of the empirical estimator of the MRLF. In the case of random censoring, Yang (1977) proposed the truncated version of the estimator for the MRLF based on the reliability estimator of Kaplan and Meier (1958) under the competing risk model in the bounded interval. Sursala and Van Ryzin (1980) studied also the nonparametric Bayesian estimator of the MRLF on a fixed interval $[0, T]$, $T < \infty$.

In real situation, it is desirable and applicable for us to investigate with more than one component. For a coherent system with m *i.i.d.* components, Doss, Freitag and Proschan (1989) proposed and studied the estimator ofthe system reliability, $\hat{R}(t)$ using Kaplan-Meier estimator of the component reliability.

In this paper, we propose an estimator $\hat{e}_T(t)$ of the MRLF for a coherent system based on the Kaplan-Meier type estimator, $\hat{R}(t)$, of the system reliability. We also investigate asymptotic properties including strong consistency and the weak convergence on a fixed interval $[0, T], T < \infty$.

2. The Proposed Estimator

Suppose that we observe a sample of n independent systems. Each system that has the same structure function ϕ is observed until it fails. For all components in each system, if the component fails before or at time of system failure, its failure times are recorded. If the component is still functioning at the time of system failure, its censoring time is recorded. By using these failure times and censoring times we want to estimate the MRLF of the system.

We define the following random variables:

$$X_{ij} = min(T_{ij}, X_i)$$

and

$$\delta_{ij} = I(T_{ij} \leq X_i) ,$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where $I(A)$ is an indicator function of a set A. Let X_{ij} denote the time on test of component j of system i and δ_{ij} be 1 if X_{ij} is uncensored and 0 if X_{ij} is censored. We assume that $X_{1j}, X_{2j}, \ldots, X_{nj}$ are *i.i.d.* with distribution $H_j(t) = 1 - R_j(t).R_\phi(t)$, for $j = 1, 2, \ldots, m$.

Remark 2.1. The sequence $(X_{ij}, \delta_{ij}, X_i), j = 1, 2, \ldots, m, i = 1, 2, \ldots, n$ contains all the information used in estimating R_ϕ and e_T. Here the random censorship model is not the pure random censoring because the censoring variables T_i depends on the component failure times, $T_{i1}, T_{i2} \ldots, T_{im}$, for $i = 1, 2, \ldots, n$. However, it is possible to regard X_i as an indirect censoring variable for some potential censoring variables which give the random censorship (Doss, Freitag and Proschan, 1989). So a parallel system is a minimum censoring case, i.e., for all i *and* j, $X_{ij} = T_{ij}$, and a series system is a maximum censoring case.

Based on the Kaplan-Meier type estimator $\hat{R}(t)$, of the system reliability, we propose an estimator $\hat{e}_T(t)$ of the MRLF $e_T(t)$ of the coherent structure ϕ of m components on $[0, X_{(n)}]$ as

$$\begin{aligned}\hat{e}_T(t) &= (\hat{R}(t))^{-1} \int_t^T \hat{R}(u)\, du \\ &= (h_\phi(\hat{R}_1(t), \hat{R}_2(t), \ldots \hat{R}_m(t))^{-1} \\ &\quad \cdot (\int_t^T h_\phi(\hat{R}_1(u), \hat{R}_2(u), \ldots, \hat{R}_m(u))\, du ,\end{aligned}$$

where $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$, $T = \inf\{t \mid F(t) = 1\}$,

and

$$\hat{R}_j(t) = \prod_{i; X_{(i)j} \le t} \left\{ \frac{n-i}{n-i+1} \right\}^{\delta_{(i)j}} \qquad j = 1, 2, \ldots, m.$$

Here $X_{(1)j} \le X_{(2)j} \le \cdots \le X_{(n)j}$ are order statistics of $X_{1j}, X_{2j}, \ldots, X_{nj}$ for $j = 1, 2, \ldots, m$, and $\hat{R}_j(t)$ is the Kaplan-Meier estimator of the reliability of component j, $j = 1, 2, \ldots, m$.

Remark 2.2. Though the MRLF $e_T(t)$ is the functional form of the system reliability and component reliabilities, it is difficult to represent the the MRLF of the system as the function of the MRLF of components. Thus we will use the informations of the system reliability and component reliabilities.

First, the strong consistency of the proposed estimator $\hat{e}_T(t)$ of $e_T(t)$ can be shown in the following theorem.

Theorem 2.1. Suppose that $R_1, R_2, \ldots, R_m$ are continuous. Let $0 < T < \infty$ be such that for $j = 1, 2, \ldots, m$, $min(R_j(T), 1 - R_j(T)) > 0$. Then

$$\sup_{0 \le t \le T} \mid \hat{e}_T(t) - e_T(t) \mid \xrightarrow{p} 0 \text{ as } n \to \infty.$$

Proof. For fixed $t \in [0, T]$,

$$\begin{aligned}
&| \hat{e}_T(t) - e_T(t) | \\
&= \left| \frac{\int_t^T \hat{R}(u)du}{\hat{R}(t)} - \frac{\int_t^T R_\phi(u)du}{R_\phi(t)} \right| \\
&= | \hat{R}(t) \cdot R_\phi(t) |^{-1} \, |R_\phi(t) \int_t^T \hat{R}(u)du - \hat{R}(t) \int_t^T R_\phi(u)du | \\
&= | \hat{R}(t) \cdot R_\phi(t) |^{-1} \, |R_\phi(t) \int_t^T \hat{R}(u)du - R_\phi(t) \int_t^T R_\phi(u)du \\
&+ R_\phi(t) \int_t^T R_\phi(u)du - \hat{R}(t) \int_t^T R_\phi(u)du | \\
&\leq | \hat{R}(t) \cdot R_\phi(t) |^{-1} (R_\phi(t) \int_t^T |\hat{R}(u) - R_\phi(u)| du \\
&+ |R_\phi(t) - \hat{R}(t)| \int_t^T R_\phi(u)du).
\end{aligned}$$

In the first term of last inequality,

$$\begin{aligned}
&\int_t^T | \hat{R}(u) - R_\phi(u) | \, du \\
&\leq \sup_{0 \leq u \leq T} \int_t^T | \hat{R}(u) - R_\phi(u) | \, du \\
&\leq \int_t^T \sup_{0 \leq u \leq T} | \hat{R}(u) - R_\phi(u) | \, du.
\end{aligned}$$

Since $\hat{R}(t)$ converges to $R_\phi(t)$ uniformly in $t \in [0, T]$ with probability 1 (Doss, Freitag and Proschan, 1989), the first term is

$$\int_t^T | \hat{R}(u) - R_\phi(u) | \, du \xrightarrow{p} 0 \ \text{ as } n \to \infty.$$

In the second term of last inequality, it is shown similarly that

$$|R_\phi(t) - \hat{R}(t)| \int_t^T R_\phi(u)du \xrightarrow{p} 0 \ \text{ as } n \to \infty,$$

since

$$|R_\phi(t) - \hat{R}(t)| \int_t^T R_\phi(u)du$$
$$\leq sup_{0\leq t\leq T}|R_\phi(t) - \hat{R}(t)| \int_t^T R_\phi(u)du.$$

This completes the proof.

3. Weak Convergency

We assume the following conditions to show the asymptotic properties of the proposed estimator.

$$(A) \quad \int_s^T \int_t^T \gamma(x,y)\; dxdy < \infty,$$

where

$$\gamma(x,y) = \sum_{j=1}^{m} I_j^*(x) I_j^*(y) \int_0^x \frac{dF_j(u)}{(1-H_j(u))R_j(u)}$$

and

$$I_j^*(\cdot) = \frac{\partial h_\phi(R_1(\cdot), R_2(\cdot), \ldots, R_m(\cdot))}{\partial R_j(\cdot)} \cdot R_j(\cdot)$$

for $0 \leq s \leq t \leq T = \inf\{t \mid F(t) = 1\}$.

We study the weak convergency of $\hat{e}_T(t)$ using the counting process theory.

Remark 3.1. To show the asymptotic distribution of the proposed estimator $\hat{e}_T(t)$, we need the following facts (Yang, 1977): Let d be the Skorohod metric on $D[0,T)$, the space of functions on the interval $[0,T)$ that are right continuous and have left limits. Define a map

$$H: D[0,T) \longrightarrow D[0,T)$$

by having

$$H(Z)(x) = R_\phi(x) \int_x^T Z(u)du - Z(x) \int_x^T R_\phi(u)du \quad \text{for } Z \in [0,T),$$

where $Z(u)$ is a mean zero Gaussian process for $\sqrt{n}(\hat{R}(t)-R_\phi(t))$ with covariance structure function

$$Cov(Z(s),Z(t)) = \sum_{j=1}^{m} I_j^*(s)I_j^*(t)\int_0^{s\wedge t}\frac{dF_j(u)}{(1-H_j(u))R_j(u)},\quad 0\le\ s,t\le\ T,$$

where

$$I_j^*(\cdot) = \frac{\partial h_\phi(R_1(\cdot),R_2(\cdot),\ldots,R_m(\cdot))}{\partial R_j(\cdot)}\cdot R_j(\cdot)$$

and $s\wedge t = \min(s,t)$.

Then H is a continuous map with respect to d. That is, the set of discontinuity points of Z has Lebesque measure zero.

The following theorem gives an important result in estimating $\hat{e}_T(t)$.

Theorem 3.1. Suppose that F, F_j and H_j, for $j = 1,\ 2,\cdots,m$ are continuous and satisfy the condition (A). Then as $n\to\infty$,

$$\sqrt{n}(\hat{e}_T(t)\ -\ e_T(t))\longrightarrow D^*(t)\ \ \textit{weakly in}\ D[0,T),$$

where $D^*(t)$ is a mean zero Gaussian process and given by

$$D^*(t) = [R_\phi(t)]^{-2}[R_\phi(t)\int_t^T Z(u)du - Z(t)\int_t^T R_\phi(u)du].$$

The covariance structure of D^* is given by

$$\begin{aligned}Cov(D^*(s),D^*(t)) =\ & [R_\phi(s)R_\phi(t)]^{-2}[R_\phi(s)R_\phi(t)E(\int_s^T\int_t^T Z(u)Z(v)dudv)\\ &+\ E(Z(s)Z(t))\int_s^T R_\phi(v)dv\int_t^T R_\phi(u)du\\ &-\ R_\phi(s)\int_t^T R_\phi(u)duE(Z(t)\int_s^T Z(v)dv)\\ &-\ R_\phi(t)\int_s^T R_\phi(u)duE(Z(s)\int_t^T Z(u)du)],\end{aligned}$$

for $0\le s\le t\le T$.

Proof. For $t < T$, one can rewrite $\sqrt{n}(\hat{e}_T(t) - e_T(t))$ as

$$\begin{aligned}
&\sqrt{n}(\hat{e}_T(t) - e_T(t)) \\
&= \sqrt{n}[\hat{R}(t)\cdot R_\phi(t)]^{-1} \, (R_\phi(t)\int_t^T \hat{R}(u)du \\
&- \hat{R}(t)\int_t^T R_\phi(u)du\ ) \\
&= \sqrt{n}[\hat{R}(t)\cdot R_\phi(t)]^{-1}(R_\phi(t)\int_t^T (\hat{R}(u) - R_\phi(u))du \\
&+ (R_\phi(t) - \hat{R}(t))\int_t^T R_\phi(u)du).
\end{aligned}$$

Since

$$\sup_{0\le t\le T} \mid \hat{R}(t) - R_\phi(t) \mid \xrightarrow{p} 0 \ \text{ as } n \to \infty,$$

the asymptotic distribution of $\sqrt{n}(\hat{e}_T(t) - e_T(t))$ is the same as that of

$$\begin{aligned}
&[R_\phi(t)]^{-2}(R_\phi(t)\int_t^T \sqrt{n}(\hat{R}(u) - R_\phi(u))du \\
&- \sqrt{n}(\hat{R}(t) - R_\phi(t))\int_t^T R_\phi(u)du).
\end{aligned}$$

Thus, the theorem follows from Theorem 2 in Doss, Freitag and Proschan (1989) and Continuity Theorem 5.1 in Billingsley (1968). Under the condition (A) the explicit form of covariance can be evaluated by the interchange of expectation with integral signs.

On the other hand, to construct confidence interval for MRLF, $e_T(t)$, we must estimate the asymptotic variance of $\hat{e}_T(t)$ given by

$$\begin{aligned}
Avar(\hat{e}_T(t)) = \Big[&E(\int_t^T Z(u)du)^2 - 2\cdot e_T(t)E(Z(t)\int_t^T Z(u)du) \\
&+ e_T{}^2(t)E(Z(t))^2\Big] \ / \ [R_\phi(t)]^2.
\end{aligned}$$

Remark 3.2. Since $E(Z(t))$ is the variance of $\sqrt{n}(\hat{R}(t) - R_\phi(t))$ it can be expressed as $\sum_{j=1}^m (I_j^*(t))^2 V_j(t)$, where $V_j(t)$ is $\dfrac{1}{n}\displaystyle\int_0^x \frac{dF_j(u)}{(1 - H_j(u))R_j(u)}$ and $I_j^*(t)$ is $\dfrac{\partial h_\phi(R_1(t), R_2(t), \ldots, R_m(t))}{\partial R_j(t)} \cdot R_j(t)$.

Now, we propose an estimator of $Avar(\hat{e}_T(t))$ as

$$\widehat{Avar}(\hat{e}_T(t)) = \left[\widehat{E}(\int_t^T Z(u)du)^2 + e_T{}^2(t)\sum_{j=1}^{m}(\hat{I}_j^*(t))^2\hat{V}_j(t)^2\right] / [\hat{R}(t)]^2,$$

where

$$\hat{V}_j(t) = \sum_{i;x_{(i)j}\le t} \frac{\delta_{(i)j}}{(n-i+1)(n-2)}$$

and

$$\hat{I}_j^*(t) = \frac{\partial h_\phi(\hat{R}_1(t), \hat{R}_2(t), \ldots, \hat{R}_m(t))}{\partial \hat{R}_j(t)} \cdot \hat{R}_j(t).$$

Also, $\widehat{E}(\int_t^T Z(u)du)^2$ is expressed as

$$\widehat{E}(\int_t^T Z(u)du)^2 = \sum_{j=1}^{m}[\left(\int_t^T \widehat{I_j^*(y)}\, dy\right)^2 \widehat{V_j(t)} + \int_t^T\left(\int_u^T \widehat{I_j^*(y)}\, dy\right)^2 \frac{d\widehat{F_j(u)}}{(1-\widehat{H_j(u)})\widehat{R_j(u)}}].$$

Theorem 3.2. Under the conditions of Theorem 3.1, $\widehat{E}(\int_t^T Z(u)du)^2$ is a consistent estimator of $E(\int_t^T Z(u)du)^2$.

Proof. For $0 \le s \le t \le T$, one can rewrite $E(\int_t^T Z(u)du)^2$ as

$$\int_t^T\int_t^T \gamma(x,y)\, dxdy = \int_t^T\int_t^T \sum_{j=1}^{m} I_j^*(x)I_j^*(y)\int_0^x \frac{dF_j(u)}{(1-H_j(u))R_j(u)}\, dxdy$$

$$= \sum_{j=1}^{m}\int_t^T\int_t^T I_j^*(x)I_j^*(y)\int_0^x \frac{dF_j(u)}{(1-H_j(u))R_j(u)}\, dxdy,$$

where

$$I_j^*(\cdot) = \frac{\partial h_\phi(R_1(\cdot), R_2(\cdot), \ldots, R_m(\cdot))}{\partial R_j(\cdot)} \cdot R_j(\cdot).$$

By interchanging integral sign (applying Fubini's theorem),

$$\begin{aligned}
&\int_t^T \int_t^T \gamma(x,y)\, dx\, dy \\
&= \sum_{j=1}^m \int_t^T I_j^*(y)[\int_0^t \int_t^T I_j^*(x)dx \cdot \frac{dF_j(u)}{(1-H_j(u))R_j(u)} \\
&\quad + \int_t^T \int_u^T I_j^*(x)dx \frac{dF_j(u)}{(1-H_j(u))R_j(u)}]dy \\
&= \sum_{j=1}^m [\int_0^t (\int_t^T I_j^*(y)dy)^2 \cdot \frac{dF_j(u)}{(1-H_j(u))R_j(u)} \\
&\quad + \int_t^T (\int_u^T I_j^*(y)dy)^2 \cdot \frac{dF_j(u)}{(1-H_j(u))R_j(u)}] \\
&= \sum_{j=1}^m [(\int_t^T I_j^*(y)dy)^2 \cdot V_j(t) + \int_t^T (\int_u^T I_j^*(y)dy)^2 \cdot \frac{dF_j(u)}{(1-H_j(u))R_j(u)}].
\end{aligned}$$

Since $\widehat{V_j(t)}$, $\hat{I}_j^*(t)$ and $\hat{R}_j(t)$ is consistent estimators of $V_j(t)$, $I_j^*(t)$ and $R_j(t)$, respectively, and $I_j^*(t)$ is continuous function, the theorem follows.

Theorem 3.3. Assume that the condition (A) is satisfied, Then for any given $t \in [0,T]$,

$$\widehat{Avar}(\hat{e}_T(t)) \xrightarrow{p} Avar(\hat{e}_T(t)) \quad \text{as } n \to \infty.$$

Proof. Since $E(Z(t)\int_t^T Z(u)du)$ converges to 0 a.s., from Theorem 3.2 and the strong consistency of $\hat{R}(t)$, this completes the proof.

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