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# EXAMPLES OF HOLOMORPHIC SIEGEL MODULAR FORMS OF WEIGHT 3/2

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In [K] Kudla has generalized the Hecke's construction [He] to produce holomorphic Siegel modular forms of genus $n$ and weight $\frac{1}{2}(n+1)$ as an integral of a non-holomorphic theta-series. Even though the existence of nonzero such forms was proved, no concrete examples were given. In this paper we consider the case $n = 2$ and construct non-trivial examples corresponding to each prime $p \equiv 3 \pmod 4$.

## 1. Introduction

Let $V$ be a $\mathbf{Q}$ vector space with non-degenerate symmetric $\mathbf{Q}$-bilinear form $( \, , \, )$ of signature $(n, 1)$. Let $L_0 \subset V$ be a lattice such that $L_0 \subset L_0^*$, where

$$L_0^* = \{v \in V | (v, v') \in \mathbf{Z} \text{ for all } v' \in L_0\}$$

is the dual lattice. Let $L = L_0^n$ and $L^* = (L_0^*)^n$. Let $G = SO(V)$ viewed as an algebraic group defined over $\mathbf{Q}$. Let

$$E_L = \{g \in G(\mathbf{Q}) | gL_0 = L_0 \text{ and } g \text{ acts trivially in } L_0^*/L_0\}$$

be the group of units and let

$$E = E_L^+ = E_L \cap G(\mathbf{R})^0,$$

where $G(\mathbf{R})^0$ is the connected component of identity. Let

$$D = \{Z \in V(\mathbf{R}) | (Z, Z) = -1\}^0$$

be one component of the hyperbolid of two sheets in $V(\mathbf{R})$, and we will identify the tangent space $T_Z(D)$ to $D$ at $Z$ with $Z^\perp$. We fix an orientation

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of $V(\mathbf{R})$ and determine an orientation of $T_Z(D)$ such that for every properly oriented basis $\{w_1, w_2\}$ for $T_Z(D)$ the basis $\{w_1, w_2, Z\}$ is properly oriented for $V$. Note that $G(\mathbf{R})^0$ preserves the orientation.

Let $X \in V(\mathbf{R})^n$ such that $(X, X) > 0$. Then $\text{span}(X)^\perp$ intersects with $D$ at a single point, say $Z_0$. Define

$$\epsilon(X) = \begin{cases} 1 & \text{if } X \text{ is properly oriented basis for } T_{z_0}(D) \\ -1 & \text{otherwise} \end{cases}$$

An element $h \in L^*/L$ is called *non-singular* if, for any choice of representative $h' \in L^*$ for $h$, the following conditions hold:

(1) $(h', h') \notin M_n(\mathbf{Z})$.

(2) If $M$ is the least positive integer such that $M(h', h') \in M_n(\mathbf{Z})$ then $M(h', h')$ is invertible modulo $M$.

For $X \in M_n(\mathbf{C})$, we let $e_n(X) = e^{i\pi tr(X)}$. If $X \in V^k$ and $Y \in V^\ell$, then we let $(X, Y) = ((X_i, Y_j)) \in M_{k,\ell}(\mathbf{Q})$. The main result of [K] is then the following.

**Theorem 1.1**. *For non-singular $h \in L^*/L$ and for $\tau \in \mathcal{H}_n$, the Siegel space of genus $n$,*

$$\vartheta(\tau, h, L, (\ ,\ )) = \sum_{\substack{X-h\in L \\ (X,X)>0,\ mod\ E}} \epsilon(X)|E_X|^{-1} e_n(\tau(X, X))$$

*is a holomorphic Siegel modular form of weight $\frac{1}{2}(n+1)$.*

In this paper, we take $V$ to be the space of binary quadratic forms

$$[a, b, c](x, y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

over $\mathbf{Q}$. We choose the standard basis for $\mathbf{Q}$ and fix isomorphisms $V \simeq \mathbf{Q}^3 \simeq S_2(\mathbf{Q})$ as follows:

$$\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2 \mapsto [a, b, c] \mapsto \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Here we view $\mathbf{Q}^3$ as a space of row vectors. We will use any of these isomorphisms to represent an element of $V$, whichever is proper at the given situation.

Define a form $(\ ,\ )$ on $V$ of signature $(2, 1)$ by

$$(X, X) = (\text{ discriminant of } X) = b^2 - ac = -det(X)$$

where $X = [a,b,c] \in V$. We begin with the lattice $L_0 = 2\mathbf{Z} \times \mathbf{Z} \times 2\mathbf{Z}$ so that $L_0^* = \mathbf{Z}^3$. Then for any positive integer $M$, we let

$$(X,Y)_M = M^{-1}(X,Y), \quad L_{0,M} = ML_0.$$

Note that the dual lattice to $L_{0,M}$ with respect to $(\ ,\ )_M$ is again $L_0^* = \mathbf{Z}^3$. Let $L = L_0^2$. For $h \in L^*/ML$, non-singular, let

$$\begin{aligned} \vartheta(\tau, h, M, L) &= \vartheta(\tau, h, L_{0,M}, (\ ,\ )_M) \\ &= \sum_{\substack{X \equiv h(ML) \\ (X,X)>0, \bmod E}} \epsilon(X)|E_X|^{-1} e_n(\tau(X,X)). \end{aligned}$$

## 2. Groups of units

We will continue to use the notations as in §1 so that we have $G = SO(2,1)$. $G(\mathbf{R})$ has two connected components

$$G(\mathbf{R}) = G(\mathbf{R})^0 \cup \lambda G(\mathbf{R})^0,$$

where $\lambda \in G$ such that $[a,b,c]\lambda = [-a,b,-c]$. We have the spinor norm map

$$\theta : G(\mathbf{R}) \to \mathbf{R}^\times/(\mathbf{R}^\times)^2$$

and that $ker\theta = G(\mathbf{R})^0$.

The group $SL_2(\mathbf{R})$ acts on $V(\mathbf{R})$ to the right via

$$[a,b,c] \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x,y) = [a,b,c](\alpha x + \beta y, \gamma x + \delta y).$$

See [GKZ]. If we use the isomorphism $V(\mathbf{R}) \simeq S_2(\mathbf{R})$, this action is stated as

$$X \cdot g = {}^t g X g$$

for $g \in SL_2(\mathbf{R})$ and $X \in S_2(\mathbf{R})$. This action preserves the form $(\ ,\ )$ and we have an exact sequence

$$1 \to \{\pm 1\} \to SL_2(\mathbf{R}) \to G(\mathbf{R})^0 \to 1.$$

In particular, we have

$$G(\mathbf{R})^0 \simeq SL_2(\mathbf{R})/\pm 1 = PSL_2(\mathbf{R}).$$

To simplify the notation, we will frequently view an element $g \in SL_2(\mathbf{R})$ as an element of $G(\mathbf{R})$. Now let

$$G_L = \{g \in G(\mathbf{R}) | Lg = L\} = \{g \in G(\mathbf{R}) | L_0 g = L_0\}$$

be the stabilizer of the lattice $L$ in $G(\mathbf{R})$.

**Lemma 2.1**. $G_L \cap G(\mathbf{R})^0 = PSL_2(\mathbf{Z})$.

*Proof.* Suppose $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_L \cap G(\mathbf{R})^0$. Then

$$\alpha^2, \alpha\gamma, \gamma^2, 2\alpha\beta, \alpha\delta + \beta\gamma, 2\gamma\delta, \beta^2, \beta\delta, \delta^2 \in \mathbf{Z}.$$

If any of $\alpha, \beta, \gamma, \delta$ is 0, then it is clear that $g \in SL_2(\mathbf{Z})$. So suppose none of these are 0. Then write

$$\alpha = k\sqrt{p}, \beta = \ell\sqrt{q}, \gamma = m\sqrt{r}, \text{ and } \delta = n\sqrt{s},$$

where $p, q, r,$ and $s$ are square-free integers. Note that $\alpha\beta, \gamma\delta$ must be integers. Since $\alpha\beta$ is an integer, we see that $p = q$. Similarly $p = r = s$. Since $1 = \alpha\delta - \beta\gamma = \sqrt{p}(kn - \ell m)$, we have $p = 1$. Hence $g \in PSL_2(\mathbf{Z})$.

**Corollary 2.2**. $G_L = PSL_2(\mathbf{Z}) \cup \lambda PSL_2(\mathbf{Z})$.

As in §1, we let $M > 0$ be an integer, and let $(X, Y) = M^{-1}(X, Y)$. Let $L_{0,M} = ML_0$, $L_M = (L_{0,M})^2$. Let

$$E_L(M) = \{g \in G_L | g \text{ acts trivially in } L_0^*/ML_0\}$$

and let $E(M) = E_L(M) \cap G(\mathbf{R})^0$. Observe that if $M|N$, then $E(M) \supset E(N)$. As usual, let

$$\Gamma(M) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbf{Z}) | \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{M} \right\}.$$

For any subgroup $\Gamma$ $SL_2(\mathbf{Z})$, we let $\bar{\Gamma} = (\Gamma \cup -\Gamma)/\pm 1 \subset PSL_2(\mathbf{Z})$. Recall that $SL_2(\mathbf{Z})/\Gamma(M) \simeq SL_2(\mathbf{Z}/M\mathbf{Z})$, and $|SL_2(\mathbf{Z}/M\mathbf{Z})| = M^3 \prod_{p|M}(1 - \frac{1}{p^2})$. The crucial fact to our construction is the following.

**Proposition 2.3**. *For any odd prime $p$, we have*

$$E(p) = \bar{\Gamma}(2p).$$

*Proof.* Let $F(p)$ be the full inverse image of $E(p)$ under the map $SL_2(\mathbf{Z}) \to PSL_2(\mathbf{Z})$. It is not difficult to show that

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in F(p) \Leftrightarrow \alpha^2 \equiv 1, \quad \beta, \gamma \equiv 0 (\bmod\ 2p).$$

Thus we have a homomorphism

$$F(p) \to (\mathbf{Z}/2p\mathbf{Z})_2^\times \simeq \{\pm 1\}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \alpha,$$

where the subscript 2 denotes the 2-torsion subgroup. This map is onto and has the kernel $\Gamma(2p)$. Thus $F(p)/\Gamma(2p) \simeq \{\pm 1\}$. The Proposition now follows.

## 3. Examples

We take $D$ as in §1 such that $1_2 \in D$. First we determine the fomula for $\epsilon$.

**Lemma 3.1.** *Let $X = (X_1, X_2)$ such that $(X, X) > 0$. If $X_i = [a_i, b_i, c_i]$, then*

$$\epsilon(X) = sgn(b_1c_2 - b_2c_1).$$

*Proof.* Note that $[x, y, z]$ is orthogonal to $[a, b, c]$ if and only if $yb - \frac{1}{2}xc - \frac{1}{2}za = 0$ if and only if $[x, y, z]$ is Euclidean orthogonal to $[-\frac{1}{2}c, b, -\frac{1}{2}a]$. Hence one vector which is orthogonal to $X_1, X_2$ is

$$\begin{aligned} Z &= [-\frac{1}{2}c_1, b_1, -\frac{1}{2}c_1] \times [-\frac{1}{2}c_2, b_2, -\frac{1}{2}c_2] \\ &= \frac{1}{2}[a_1b_2 - b_1a_2, \frac{1}{2}(a_1c_2 - c_1a_2), b_1c_2 - c_1b_2]. \end{aligned}$$

Note that $(Z, Z) < 0$ since $Z$ is orthogonal to span$(X)$. Now $Z_0 = \alpha$ sgn$(b_1c_2 - c_1b_2)Z$ for some $\alpha > 0$. It is clear that $\epsilon(X) = \text{sgn}(\det(X, \text{sgn}(b_1c_2 - c_1b_2)Z))$. But $\det(X, \text{sgn}(b_1c_2 - c_1b_2)Z) = \text{sgn}(b_1c_2 - c_1b_2)(-(Z, Z))$. Hence $\epsilon(X) = \text{sgn}(b_1c_2 - c_1b_2)$.

Note that $E(M)_X = 1$. This follows from the fact that if $U$ is a regular hyper plane of $V$, and $\sigma$ is an isometry of $U$ into $V$, then there are exactly

2 extensions of $\sigma$ to $O(V)$ and one is a symmetry times the other [O]. So $E(M)_X \subset \{e, \tau\}$, where $\tau$ is the symmetry with respect to the line orthogonal to the hyperplane span$(X)$. But $\theta(\tau) = 1$, where $\theta$ denotes the spinor norm as before. Thus we have $E(M)_X = 1$.

Next observe that

$$\vartheta(\tau, hg, M) = \theta(g)\vartheta(\tau, h, M) \tag{3.1}$$

for $g \in G_L$ and

$$\vartheta(\tau, h, M) = \theta(g)\vartheta(\tau, h, M) \tag{3.2}$$

for $g \in E_L(M)$. Recall that $\lambda PSL(2, \mathbf{Z}) \subset G_L$ and $\theta(\lambda) = -1$.

**Proposition 3.2.** *If all prime factors of $M$ are congruent to 1 mod 4, then $\vartheta(\tau, h, M)$ is identically 0.*

*Proof.* For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbf{Z}), \lambda g \in E_L(M)$ if and only if

$$\alpha^2 \equiv -1 \ (\text{mod } 2M), \quad \beta \equiv \gamma \equiv 0 \text{ mod } M'), \tag{3.3}$$

where $M'$ =lcm$(2, M)$. By assumption, $x^2 = -1$ has a solution $\alpha$ mod $2M$. Note that $-\alpha$ is a solution, also. Take $\delta = -\alpha$, and $\beta = \gamma = 0$. This choice of $\alpha$, $\beta$, $\gamma$, $\delta$ satisfies (3.3) and $\alpha\delta - \beta\gamma = -\alpha^2 \equiv 1$ mod $2M$. Since $SL_2(\mathbf{Z})/\Gamma(2M) \simeq SL_2(\mathbf{Z}/2M\mathbf{Z})$, there exists a $g \in SL_2(\mathbf{Z})$, $g$ congruent to $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ mod $2M$ so that $\lambda g \in E_L(M)$. The proposition now follows from (3.2).

Now let $M = p \equiv 3$ (mod 4) be a prime so that $-1$ is a nonsquare mod $p$. Take $e_1 = [0, 1, 0]$, $e_2 = [1, 0, -1] \in V$, and let $h_0 = [e_1, e_2]$. Let

$$L_1 = \{X \in L^* | (X, X) = (h_0, h_0), X \equiv h_0(pL)\}.$$

**Lemma 3.3.** $L_1 = h_0\Gamma(p)$.

*Proof.* For any $X \in L_1$, there exists an element $g \in G(\mathbf{Q})$ such that $h_0 g = X$. Since $G(\mathbf{R}) = PSL_2(\mathbf{R}) \cup \lambda PSL_2(\mathbf{R})$, there exists an element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL_2(\mathbf{R})$ such that $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ or $g = \lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Suppose $g = \lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then $h_0 g \equiv h_0 \pmod{pL}$ implies that

$$2\alpha\gamma \equiv 0 \pmod{2p}, \alpha\delta + \beta\gamma \equiv 1 \pmod{p}, 2\beta\delta \equiv 0 \pmod{2p},$$

$$-\alpha^2 + \gamma^2 \equiv 1 \pmod{2p}, -\alpha\beta + \gamma\delta \equiv 0 \pmod{p}, -\beta^2 + \delta^2 \equiv -1 \pmod{2p},$$

Since $\alpha\delta - \beta\gamma = 1$, we have $\alpha\delta \equiv 1 \pmod{p}$, $\beta\gamma \equiv 0 \pmod{p}$. Thus $\alpha^2, \beta^2, \gamma^2, \delta^2$ and $\alpha\beta, \beta\gamma, \gamma\delta, \delta\alpha$ are all integers. So $\alpha, \beta, \gamma, \delta \in \mathbf{Z}$. Since $\alpha\delta \equiv -1 \pmod{p}$ and $\beta\delta \equiv 0 \pmod{p}$, we have $\beta \equiv 0 \pmod{p}$. But then $\delta^2 \equiv -1 \pmod{p}$, which is impossible. Hence $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $g \in PSL_2(\mathbf{Z})$ by the similar argument as above. Now $h_0 g \equiv h_0 \pmod{pL}$ if and only if

$$\alpha\gamma \equiv 0 \pmod{p}, \alpha\delta + \beta\gamma \equiv 1 \pmod{p}, \beta\delta \equiv 0 \pmod{p},$$

$$\alpha^2 - \gamma^2 \equiv 1 \pmod{2p}, \alpha\beta - \gamma\delta \equiv 0 \pmod{p}, \beta^2 - \delta^2 \equiv -1 \pmod{2p}.$$

Since $\alpha\delta - \beta\gamma = 1$, we have $\alpha^2 - \gamma^2 \equiv \beta^2 - \delta^2 \equiv 1 \pmod{2}$. Hence $h_0 g \equiv h_0 \pmod{pL}$ if and only if

$$\alpha\gamma \equiv 0 \pmod{p}, \alpha\delta + \beta\gamma \equiv 1 \pmod{p}, \beta\delta \equiv 0 \pmod{p},$$

$$\alpha^2 - \gamma^2 \equiv 1 \pmod{p}, \alpha\beta - \gamma\delta \equiv 0 \pmod{p}, \beta^2 - \delta^2 \equiv -1 \pmod{p}.$$

Again, $\alpha\delta - \beta\gamma = 1$ shows that $\alpha\delta \equiv 1 \pmod{p}$. Hence $\beta \equiv \gamma \equiv 0 \pmod{p}$ and $\alpha \equiv \delta \equiv \pm 1 \pmod{p}$. Therefore $g \in \Gamma(p)$. The inverse inclusion is clear.

**Theorem 3.4.** *Let* $e_1 = [0, 1, 0], e_2 = [1, 0, -1] \in V$, *and* $h_0 = [e_1, e_2]$. *Then for any prime* $p \equiv 3 \pmod{4}$, *we have* $\vartheta(\tau, h_0, p) \neq 0$.

*Proof.* Let $\vartheta(\tau, h_0, p) = \sum_{g>0} a(g) e(\tau g)$ be the Fourier expansion of $\vartheta$. Then, by the Proposition 2.3 and Lemma 3.3, we have

$$a(1_2) = \sum_{\substack{X \in L_1 \\ \bmod \Gamma(2p)}} \epsilon(X) = \sum_{g_1 \in \Gamma(p)/\Gamma(2p)} \epsilon(h_0 g_1) = -|\Gamma(p)/\Gamma(2p)| = -6.$$

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