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ON A STRUCTURE SATISFYING AN ALGEBRAIC EQUATION $\overline{\overline{X}} = a^2X + \sum_{p=1}^{r} A_p(X)T^p$

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Differentiable manifolds with almost contact structures were investigated by W. M. Boothby - H. C. Wang [1], D. E. Blair [2], S. I. Goldberg - K. Yano [4], and among others. S. Sasaki [3] defined the notion of (ϕ, ξ, η, g)-structure on a differentiable manifold and showed that the structure is closely related to the almost contact structure. The purpose of this paper is to study a manifold with differentiable structure defined by an algebraic equation $\overline{\overline{X}} = a^2X + \sum_{p=1}^{r} A_p(X)T^p$ and obtain its integrability conditions. In particular this manifold reduces to an almost r-contact hyperbolic manifold.

The results of this paper have been announced by the author in Abstracts, American Mathematical Society [8].

1. Introduction

Let us consider an n-dimensional $(n = m + r)$ real differentiable manifold M^n of class C^∞. Let there exist a C^∞ function F, $r(C^\infty)$ contravariant vector fields $T^1, T^2, \cdots, T^r$ and $r(C^\infty)$ 1-forms $A_1, A_2, \cdots, A_r$ satisfying the following conditions:

$$\overline{\overline{X}} = a^2X + \sum_{p=1}^{r} A_p(X)T^p, \tag{1.1}$$

where a is any nonzero complex number. Let

$$\overline{X} = F(X), \tag{1.2}$$

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(1.3) $$\overline{T^p} = 0, \text{ for } p = 1, 2, \cdots, r$$

(1.4) $$A_p\overline{X} = 0, \text{ for an arbitrary vector field } X$$

(1.5) $$A_pT^p = -a^2,$$

Let M^n be endowed with the Riemannian metric tensor g such that

(1.6) $$A_qX \overset{\text{def}}{=} g(T^q, X), \text{ for } q = 1, 2, \cdots, r.$$

(1.7) $$g(\overline{X}, \overline{Y}) = -a^2 g(X,Y) - \sum_{p=1}^{r} A_p(X)A_p(Y),$$

where g is a nonsingular metric tensor.

We suppose that F gives to M^n a differentiable structure defined by an algebraic equation (1.1). It is well known that a manifold is an almost r-contact metric manifold if it is of dimension $2n + r$ and $a = \pm i$.

Let us define

(1.8) $$'F(X,Y) \overset{\text{def}}{=} g(X, \overline{Y})$$

Barring X in (1.8) we obtain

(1.9) $$'F(X,Y) = g(\overline{\overline{X}}, \overline{Y})$$

which in view of (1.1) and (1.6), yields

(1.10) $$'F(\overline{X}, Y) = a^2 g(X,Y) + \sum_{p=1}^{r} A_p(X)A_p(Y),$$

Barring Y in (1.8) we obtain

(1.11) $$'F(X,Y) = g(\overline{X}, \overline{Y}),$$

which with the help of (1.7) yields

(1.12) $$'F(X, \overline{Y}) = -\{a^2 g(X,Y) + \sum_{p=1}^{r} A_p(X)A_p(Y)\}.$$

Thus from (1.10) and (1.12) we get

(1.13) $$F(X, \overline{Y}) = -'F(\overline{X}, Y).$$

Replacing X by T^q in (1.8) and making use of (1.3), we get

(1.14) $$'F(T^q, Y) = 0.$$

Barring X in (1.13) and making use of (1.1) and (1.14) we get

$$'F(\overline{X},\overline{Y}) = -a^2{}'F(X,Y). \tag{1.15}$$

Also barring Y in (1.7) and with the help of (1.3) and (1.4) we get

$$g(\overline{X},Y) = -g(X,\overline{Y}). \tag{1.16}$$

Thus from (1.8) and (1.16) we have

$$'F(X,Y) = -'F(Y,X). \tag{1.17}$$

Hence $'F(X,Y)$ is skew symmetric.

2. COMPLETE INTEGRABILITY CONDITIONS OF DIFFERENTIAL MANIFOLD M^n

The Nijenhuis tensor for the $(1,1)$ tensor field F can be written as

$$N(X,Y) = [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[X,\overline{Y}]} - \overline{[\overline{X},Y]}. \tag{2.1}$$

Thus in view of (1.1), we have

$$\begin{aligned} N(X,Y) &= [\overline{X},\overline{Y}] + a^2[X,Y] + \sum_{p=1}^{r} A_p([X,Y])T^p \\ &\quad -\overline{[X,\overline{Y}]} - \overline{[\overline{X},Y]} \end{aligned} \tag{2.2}$$

Definition 2.1. The differentiable manifold M^n is completely integrable, if the Nijenhuis tensor vanishes.

Theorem 2.1. *In order that a differentiable manifold be completely integrable, it is necessary that*

$$\sum_{p=1}^{r} A_p([\overline{X},\overline{Y}])T^p = 0. \tag{2.3}$$

Proof. Barring X in (2.2) and using (1.1) we get

$$\begin{aligned} N(\overline{X},Y) &= a^2[X,\overline{Y}] + \sum_{p=1}^{r} A_p(X)[T^p,\overline{Y}] + a^2[\overline{X},Y] \\ &\quad + \sum_{p=1}^{r} A_p([\overline{X},Y])T^p - \overline{[\overline{X},\overline{Y}]} - a^2\overline{[X,Y]} \\ &\quad - \sum_{p=1}^{r} A_p(X)\overline{[T^p,Y]}. \end{aligned} \tag{2.4}$$

Now barring the whole equation (2.4) and making use of (1.1), we obtain

$$(2.5)\quad \begin{aligned}\overline{N(\overline{X},Y)} &= a^2\overline{[X,\overline{Y}]}+\sum_{p=1}^{r}A_p(X)\overline{[T^p,\overline{Y}]}+a^2\overline{[\overline{X},Y]}\\ &\quad -a^2[\overline{X},\overline{Y}]-\sum_{p=1}^{r}A_p([\overline{X},\overline{Y}])T^p-a^4[X,Y]\\ &\quad -a^2\sum_{p=1}^{r}A_p([X,Y])T^p-a^2\sum_{p=1}^{r}A_p(X)[T^p,Y]\\ &\quad -\sum_{p,q=1}^{r}A_p(X)A_q([T^p,Y])T^q\end{aligned}$$

In consequence of equations (2.4) and (2.5) we have

$$(2.6)\quad \begin{aligned}\overline{N(\overline{X},Y)}+a^2N[X,Y] &= \sum_{p=1}^{r}A_p(X)\overline{[T^p,\overline{Y}]}-\sum_{p=1}^{r}A_p([\overline{X},\overline{Y}])T^p\\ &\quad -a^2\sum_{p=1}^{r}A_p(X)[T^p,Y]-\sum_{p,q=1}^{r}A_p(X)A_q([T^p,Y])T^q.\end{aligned}$$

Now in view of the equation

$$(2.7)\qquad N(T^p,Y)=a^2[T^p,Y]+\sum_{p=1}^{r}A_p(X)[T^p,Y]T^p-\overline{[T^p,\overline{Y}]}.$$

and (2.6) we obtain

$$(2.8)\qquad \begin{aligned}\overline{N(\overline{X},Y)}+a^2N(X,Y) &= -\sum_{p=1}^{r}A_p(X)\{N(T^p,Y)\}\\ &\quad -\sum_{p=1}^{r}A_p([X,Y])T^p\end{aligned}$$

For the complete integrabilty of the manifold M^n, the equation (2.8) reduces to (2.3).

Theorem 2.2. *For a completely integrable manifold M^n, we have*

$$(2.9)\qquad \begin{aligned}&\sum_{p=1}^{r}A_p(X)\{[T^p,\overline{Y}]-\overline{[T^p,Y]}\}+\sum_{p=1}^{r}A_p([\overline{X},Y])T^p\\ &=\sum_{p=1}^{r}A_p(Y)\{[\overline{X},T^p]-\overline{[X,T^p]}\}+\sum_{p=1}^{r}A_p([X,\overline{Y}])T^p.\end{aligned}$$

Proof. Barring X and Y in (2.4) and using (1.1), we obtain respectively the following

$$(2.10)\qquad \begin{aligned} N(\overline{X},Y) &= a^2[X,\overline{Y}] + \sum_{p=1}^{r} A_p(X)[T^p,\overline{Y}] + a^2[\overline{X},Y] \\ &\quad + \sum_{p=1}^{r} A_p([\overline{X},Y])T^p - \overline{[\overline{X},\overline{Y}]} - a^2\overline{[X,Y]} \\ &\quad - \sum_{p=1}^{r} A_p(X)\overline{[T^p,Y]}. \end{aligned}$$

and

$$(2.11)\qquad \begin{aligned} N(X,\overline{Y}) &= a^2[\overline{X},Y] + \sum_{p=1}^{r} A_p(Y)([\overline{X},T^p]) + a^2[X,\overline{Y}] \\ &\quad + \sum_{p=1}^{r} A_p([X,\overline{Y}])T^p - \overline{[\overline{X},\overline{Y}]} - \sum_{p=1}^{r} A_p(X)\overline{[X,T^p]} - \overline{[X,\overline{Y}]}. \end{aligned}$$

Thus from (2.10) and (2.11), we have

$$(2.12)\qquad \begin{aligned} N(\overline{X},Y) - N(X,\overline{Y}) &= \sum_{p=1}^{r} A_p(X)\{[T^p,\overline{Y}] - \overline{[T^p,Y]}\} \\ &\quad + \sum_{p=1}^{r} A_p([\overline{X},Y])T^p - \sum_{p=1}^{r} A_p([X,\overline{Y}])T^p \\ &\quad - \sum_{p=1}^{r} A_p(Y)\{[\overline{X},T^p] - \overline{[X,T^p]}\} \end{aligned}$$

Now putting $N(X,Y) = 0$ in (2.12) we obtain (2.9).

3. NON UNIQUENESS OF THE ALGEBRAIC EQUATION

In this section we take C^∞ manifold M^n admitting a C^∞ tensor field f of the type $(1,1)$, $r(C^\infty)$ 1-forms $'A_1,' A_2,' A_3, \cdots ,' A_r$ and C^∞ contravariant vector fields $'T^1,' T^2, \cdots ,' T^r$ and we define the following relations:

$$(3.1)\qquad \mu(f(X)) \overset{\text{def}}{=} \overline{\mu(X)} - \sum_{p=1}^{r} \alpha(X)T^p,$$

$$T^p \overset{\text{def}}{=} \mu('T^p), \text{ for } p = 1, 2, \cdots, r, \tag{3.2}$$

$$'A_p(X) \overset{\text{def}}{=} A_p(\mu(X)) - \alpha(f(X)). \tag{3.3}$$

where α is some scalar function and μ is a nonsingular vector valued function.

Theorem 3.1. *In a differentiable manifold the algebraic equation defined by*

$$F^2(X) = a^2 X + \sum_{p=1}^{r} A_p(X) T^p,$$

is not unique, if and only if (3.2) and (3.3) hold.

Proof. Putting $f(X)$ for X in (3.1) and making use of (1.1) and (3.1) we get

$$\begin{aligned}\mu(f(f(X))) &= \overline{\mu f(X)} - \sum_{p=1}^{r} \alpha(f(X)) T^p, \\ &= a^2 \mu(X) + \sum_{p=1}^{r} A_p(\mu(X)) T^p - \sum_{p=1}^{r} \alpha(X) \overline{T}^p, \\ &\quad - \sum_{p=1}^{r} \alpha(f(X)) T^p,\end{aligned}$$

which in view of (1.3) yields

$$\mu(f(f(X))) = a^2 \mu(X) + \sum_{p=1}^{r} A_p(\mu(X)) T^p - \sum_{p=1}^{r} \alpha(f(X)) T^p.$$

Since μ is a nonsingular vector valued linear function, thus making use of (3.2) and (3.3) we obtain

$$f(f(X)) = a^2 X + \sum_{p=1}^{r} \{A_p(\mu(X)) - \alpha(f(X))\}' T^p.$$

or

$$f(f(X)) = a^2 X + \sum_{p=1}^{r} {}'A_p(X)' T^p.$$

Therefore, the algebraic equation defined by (1.1) is not unique.

Theorem 3.2. *Let there be two algebraic equations satisfying (1.1) in M^n and related by (3.1) then we have*

$$a^2 \alpha(X) = A_p \mu(f(X)), \tag{3.4}$$

$$\alpha('T^p) = 0, \tag{3.5}$$

and

$$\alpha \neq' A_p. \tag{3.6}$$

Proof. The proof of (3.4) follows in consequence of (3.3) and (1.1). Putting $'T^p$ for X in (3.4) we at once get (3.5). (3.6) follows immediately after putting $'A_p$ for α in (3.5), thus giving $'A_p('T^p) = 0$, which is not true because of (1.5) hence $\alpha \neq' A_p$.

References

[1] W. M. Boothby and H. C. Wang, *On contact manifolds*, Ann. of Math., 68(1958), 721-734.

[2] D. E. Blair, *The theory of quasi-Sasakian structures*, Journal of Differential Geometry; 1(1967), 331-345.

[3] S. Sasaki and Y. Hatakeyama, *Differentiable manifolds with certain structures which are closely related to almost contact structure II*, Tohoku Math. J., Vol. 13 (1961), 281-294.

[4] S. I. Goldberg and K. Yano, *Noninvariant hypersurfaces of almost contact manifolds*, J. Math. Society, Japan; Vol. 22 (1970), 25-34.

[5] M. Prvanovic, *Holomorphically projective transformation in a locally product space*, Mathematica Balkanica, Vol. 1 (1971), 195-213.

[6] K. Yano, *Differential Geometry on complex and almost complex spaces*, Pergamon Press, New York (1965).

[7] A. Gray, *Some examples of almost Hermitian manifolds*, Illinoios J. Math, 10(1966), 353-366.

[8] L. Das, *On a structure satisfying an algebraic equation* $\overline{\overline{X}} = a^2X + \sum_{p=1}^{r} A_p(X)T^p$, Abstracts American Mathematical Society; Vol.II(1990), 513-514.

[9] L. Das, *On derivations with almost r-contact metric manifold*, Publications Mathematicae Debrecan, Hungary, Vol. 35(1988).

[10] L. Das, *On almost r-contact metric manifold*, Comptes Rendus del' Academie Bulgare des Sciences, Vol. 32 (1979) 11-714.

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