GENERATING CELLULAR DECOMPOSITIONS OF $E^3$ AND THE NONEXISTENCE OF CERTAIN FINITE-TO-ONE MAPPINGS

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1. Introduction

Some years ago, Hurewicz constructed monotone mappings $m$ of compacta in $E^3$ onto given compacta $Y$. The sets $m^{-1}m(x)$ for $x \in E^3$ are, of course, compact and connected, but without various other connectivity properties ($LC^n, lc^n, n - LC$, etc., for $n > 0$). It is difficult to use Hurewicz's technique (and, indeed, generally impossible) to construct cellular mappings from compacta in $E^3$ onto certain 2-dimensional polyhedra.

We shall give conditions under which Hurewicz's technique can be modified to yield very nice cellular decompositions of $E^3$ (closed mappings $f$ defined on $E^3$ with $f^{-1}f(x)$ cellular for each $x \in E^3$). It will be shown that it is impossible to use his technique (in some sense) to obtain certain special cellular decompositions of $E^3$. A surprising consequence is that certain finite-to-one mapping from the Cantor Space onto any $n$-cell do not exist.

Some interesting general questions arise.

Suppose that $K^n$ is an $n$-dimensional polyhedron (more generally, an $n$-dimensional compactum). When does there exists a metric space $(Y, d)$ and a closed cellular mapping $f$ of $E^3$ onto $Y$ such that $Y$ contains a homeomorphic copy of $K^n$? What is the least natural number $m$ such that for each metric space $(Y, d)$ and each closed cellular mapping $f$ of $E^3$ onto $Y$, $E^m$ contains a homeomorphic copy of $Y$? Some related work is contained in [1; 2; 3; 4; 5].
2. Finite-to-one mappings on the Cantor Space

The existence of certain finite-to-one (continuous) mappings defined on the usual Cantor Space (contained in the closed interval \([0, 1]\)) imply the existence of very nice cellular decompositions of \(E^3\). Consider the following.

**Definition.** Suppose that \(f\) is a continuous mapping from \(C\) (the Cantor Space) into a metric space \((X, d)\). The collection \(G_f = \{f^{-1}f(x) | x \in C\}\) is not properly situated on \(C\) if and only if for some \(g \in G_f\) containing points \(a, b\) and \(c\) with \(a < b < c\), there is a sequence \(\{g_i\}\) in \(G_f\) such that there are points \(a_i, c_i \in g_i\) with \(a_i < c_i\), \(\lim a_i = a\) and \(\lim c_i = c\), but there is no sequence \(\{b_i\}\) of points \(b_i \in g_i\) such that \(a_i < b_i < c_i\). Otherwise, \(G_f\) is properly situated on \(C\). The mapping \(f\) is said to be admissible on \(C\) iff \(G_f\) is properly situated on \(C\).

3. Generating cellular decompositions of \(E^3\) with each non-degenerate element being a simple polygonal arc

**Theorem 1.** Suppose that \(K\) is a compactum and that \(f\) is a finite-to-one admissible mapping of \(C\) onto \(K\). Then there is a cellular decomposition \(G\) of \(E^3\) such that the decomposition space \(E^3/G\) contains a homeomorphic copy of \(K\).

**Proof.** Let \(J\) be the join of the interval \(A = [0, 1]\) (containing \(C\)) with another copy \(B = [0, 1]\). Suppose that \(g \in G_f\) lies in \(A\). Let \(g'\) denote the copy of \(g\) in \(B\). Write \(g = \{p_1, p_2, \ldots, p_n\}\) where \(p_i < p_{i+1}\) for \(i = 1, 2, \ldots, n - 1\). Similarly, write \(g' = \{p'_1, p'_2, \ldots, p'_n\}\) where \(p'_i < p'_{i+1}\). Now, in the join \(J\) construct a simple polygonal arc as follows. Connect \(p_i\) to \(p'_i\) with a straight line interval with end points \(p_i\) and \(p'_i\). Also, connect \(p_i\) to \(p'_{i+1}\) with a straight line interval with these points as end points for \(i = 1, 2, \ldots, n - 1\). The union of these straight line intervals in a simple polygonal arc, which we denote \(gg'\).

Let \(G\) be the collection of all such \(gg'\) for \(g \in G_f\) and all singletons \(x\) in \(E^3\) not on one of these arcs.

It is easy to see that \(G\) is an upper semicontinuous decomposition of \(E^3\), since \(f\) is admissible on \(C\). Consequently, the quotient mapping \(p : E^3 \to E^3/G\) is a closed cellular mapping with \(p^{-1}p(x) \in G\) for each \(x \in E^3\). Clearly, \(E^3/G\) contains a homeomorphic copy of \(K\).

The dimension of \(K\) in the above theorem can not exceed three, since
Kozlowski and Walsh [7] have proved that cell-like mappings on 3-manifolds do not raise dimension.

It has been proved by Flores [5] that the complex $K^n$ consisting of all faces of dimension less than or equal to $n$ of a $(2n + 2)$-simplex cannot be embedded in $E^{2n}$. Thus, the following corollary would reduce an outstanding question to one of finding an admissible finite-to-one mapping of the Cantor Space onto the 2-skeleton $K^2$ of a 6-simplex $\sigma^6$.

**Corollary.** If there is a finite-to-one admissible mapping $f$ of $C$ onto the 2-skeleton $K^2$ of a 6-simplex $\sigma^6$, then there is a cellular upper semicontinuous decomposition $G$ of $E^3$ such that $E^3/G \times E^1$ is not homeomorphic to $E^4$.

**Proof.** Construct $G$ as in Theorem 1. If $E^3/G \times E^1$ is homeomorphic to $E^4$, then $K^2$ embeds in $E^4$ contrary to the work of Flores. Consequently, $E^3/G \times E^1$ is not homeomorphic to $E^4$.

A result of Daverman and Preston [4] states that if $G$ is a cell-like usc (upper semicontinuous) decomposition of $E^3$ such that the image under the projection mapping $p : E^3 \to E^3/G$ of the set $\mathcal{N}_G = \{x | p^{-1}p(x) \neq x\}$ has dim $\leq 1$, then $E^3/G \times E^1$ is homeomorphic to $E^4$. There are related results. However, the question of whether or not $E^3/G \times E^1$ is homeomorphic to $E^4$ for each cell-like (or cellular) decomposition $G$ of $E^3$ remains unanswered.

The following theorem states that any one-dimensional compactum has a homeomorphic copy in some nice cellular decomposition of $E^3$.

**Theorem 2.** Suppose that $K$ is a one-dimensional compactum. Then there is an usc decomposition $G$ of $E^3$ such that each non-degenerate element of $G$ is a simple polygonal arc and $E^3/G$ contains a homeomorphic copy of $K$.

**Proof.** There is a mapping $f$ of $C$ onto $K$ such that $|f^{-1}f(x)| \leq 2$ for each $x \in C$. Note that $f$ must be admissible. Construct $G$ as in the proof of Theorem 1.

Suppose that $K$ is an $n$-dimensional compactum. Then there is a mapping $f$ of $C$ onto $K$ such that $|f^{-1}f(x)| \leq n + 1$ for each $x \in C$ and the collection $G_f^{n+1} = \{f^{-1}f(x) | f^{-1}f(x) \text{ is exactly } n + 1 \text{ points} \}$ is countable. This follows easily from the covering dimension of $K$ and the 0-dimensionality of $C$. However, $G_f^{n+1}$ may not be a null collection. Recall that $G_f^{n+1}$ is null iff for each $\varepsilon > 0$, at most a finite number of the
elements of $G^n_{f+1}$ have diam $> \varepsilon$. We shall show that no such mappings $f$ exist in some situations with $G^3_f$ being a null collection. First, consider the following theorem.

**Theorem 3.** Suppose that $K$ is a compactum and that $f$ is a mapping of $C$ onto $K$ which is at most three-to-one. If $G^3_f$ is a null collection then there is a cellular usc decomposition $G$ of $E^3$ such that $E^3/G$ contains a homeomorphic copy of $K$.

**Proof.** Write $G^3_f = \{T_i\}$. Thus, diam $T_i \to 0$. For each $i$, let $T_i = \{a_i, b_i, c_i\}$, where $a_i < b_i < c_i$. Let $H_j = \{f^{-1}f(x)|f^{-1}f(x) \text{ is exactly } j \text{ points} \}$. Then $H_3 = G^3_f$.

For each $i$, construct a disk $D(T_i)$ bounded by three semi-circles with end points $a_i$ and $b_i$, $a_i$ and $c_i$, and $b_i$ and $c_i$, all lying in the half-plane containing the $x$-axis making an angle of $a_i$ with the $xy$-plane.

There exist pairwise disjoint intervals $I(a_1), I(b_1)$ and $I(c_1)$ in $[0, 1]$ such that:

1. $a_1 \in I(a_1), b_1 \in I(b_1)$ and $c_1 \in I(c_1)$;
2. $[I(a_1) \cup I(b_1) \cup I(c_1)] \cap C = A_1$ is an open and closed subset of $C$; and
3. for $i > 1$, $T_i \cap I(p) \neq \emptyset$ for at most one of $p = a_1, b_1$ or $c_1$.

Let $S_1 = \{g|g \in H_2 \cup H_3, A_1 \supset g, \text{ and } g \text{ is not contained entirely in one of } I(a_1), I(b_1) \text{ and } I(c_1)\}$. Note that $S_1^* \text{ is closed (where } M^* \text{ denotes the union of the elements of the collection } M\). Also, $T_1 \in S_1$ and $T_i \notin S_1$ for $i > 1$.

Let $h_1$ denote an order-preserving homeomorphism of $I(b_1) \cap C$ into $I(a_1)$ such that $h_1(b_1) = a_1$ and $h_1(x) \notin I(a_1) \cap C$ for $x \in I(b_1) \cap (C - b_1)$.

Now, if $g \in S_1$ and $g = \{x, y\}$, $x < y$, then construct a semi-circle $C(g)$ with end points $x$ and $y$ lying in the half-plane containing the $x$-axis and making either an angle of $x$ with the $xy$-plane if $x \in I(a_1)$ or an angle of $h_1(x)$ if $x \in I(b_1)$.

Consider $T_2 = \{a_2, b_2, c_2\}$. Construct pairwise disjoint intervals $I(a_2), I(b_2)$ and $I(c_2)$ in $[0, 1]$ such that:

1. $a_2 \in I(a_2), b_2 \in I(b_2)$ and $c_2 \in I(c_2)$;
2. $[I(a_2) \cup I(b_2) \cup I(c_2)] \cap C = A_2$ is an open and closed subset of $C$;
3. $A_2 \cap S_1^* = \emptyset$;
4. each interval has length $< 1/2^2$; and
5. if $i > 2$, then $T_i \cap I(p) \neq \emptyset$ for at most one of $p = a_2, b_2$ or $c_2$.

Let $h_2$ denote an order-preserving homeomorphic of $I(b_2) \cap C$ into $I(a_2)$ such that $h_2(b_2) = a_2$, $h_2(x) \notin I(a_2) \cap C$ for $x \in I(b_2) \cap (C - b_2)$,
and $h_2(x) \neq h_1(y)$ for any $x$ and $y$.

Let $S_2 = \{g|g \in H_2 \cup H_3, A_2 \supseteq g,\text{ and } g\text{ is not contained entirely in one of } I(a_2), I(b_2)\text{ and } I(c_2)\}$. Note that $S_2^*$ is closed. Also, $T_2 \in S_2$.

If $g \in S_2$ and $g = \{x, y\}, x < y$, then construct a semi-circle $C(g)$ with end points $x$ and $y$ lying in the half-plane containing the $x$-axis and making either an angle of $x$ with the $xy$-plane if $x \in I(a_2)$ or an angle of $h_2(x)$ if $x \in I(b_2)$.

Continuing in this manner, we construct, for each $i$, pairwise disjoint intervals $I(a_{i+1}), I(b_{i+1})$ and $I(c_{i+1})$ in $[0, 1]$ such that:

1. $a_{i+1} \in I(a_{i+1}), b_{i+1} \in I(b_{i+1})$ and $c_{i+1} \in I(c_{i+1})$;
2. $[I(a_{i+1}) \cup I(b_{i+1}) \cup I(c_{i+1})] \cap C = A_{i+1}$ is an open and closed subset of $C$;
3. $A_{i+1} \cap \bigcup_{k=1}^{i} S_k^* = \emptyset$, where $S_k = \{g|g \in H_2 \cup H_3, A_k \supseteq g,\text{ and } g\text{ is not contained entirely in one of } I(a_k), I(b_k)\text{ and } I(c_k)\}$;
4. each interval has length $< 1/2^{i+1}$; and
5. if $k > i + 1$, then $T_k \cap I(p) \neq \emptyset$ for at most one of $p = a_{i+1}, b_{i+1}$ or $c_{i+1}$.

For each $i$, there is an order-preserving homeomorphism $h_i$ of $I(b_i) \cap C$ into $I(a_i)$ such that:

1. $h_i(b_i) = a_i$;
2. $h_i(x) \not\in I(a_i) \cap C$ for $x \in I(b_i) \cap (C - b_i)$, and
3. $h_i(x) \neq h_j(y)$ for any $x$ and $y$ for $i \neq j$.

Note that $S_i^*$ is closed and $T_i \in S_i$.

If $g \in S_i$ and $g = \{x, y\}, x < y$, then construct a semi-circle $C(g)$ with end points $x$ and $y$ lying in the half-plane containing the $x$-axis and making either an angle of $x$ with the $xy$-plane if $x \in I(a_i)$ or an angle of $h_i(x)$ if $x \in I(b_i)$.

If $\{x, y\} = g \in H_2$ where $g \not\in S_i$ for any $i$, then construct a semi-circle $C(x, y)$ in the half-plane containing the $x$-axis and making an angle of $-x$ with the $xy$-plane. Thus, $C(x, y)$ lies below the $xy$-plane whereas the other disks and semi-circles constructed thus far lie above the $xy$-plane.

Let $H$ denote the collection of the various disks and semi-circles constructed in this manner. Let $G$ denote the collection consisting of the elements of $H$ together with the singletons in $E^3$ which do not belong to an element of $H$. It follows that $G$ is a cellular (point-like) used decomposition of $E^3$. Furthermore, the decomposition space $E^3/G$ contains a homeomorphic copy of $K$.

The proof of Theorem 3 can be easily adapted to prove the following
Theorem 4. Suppose that $f$ is an at most three-to-one mapping of the Cantor Space $C$ onto a compactum $K$. Furthermore, $G^3_f = \{\{a_k, b_k, c_k\}|k = 1, 2, 3, \ldots\}$ is countable, and, for each $i$ and each $\varepsilon > 0$, there exist pairwise disjoint closed intervals $I(a_i), I(b_i)$ and $I(c_i)$ containing $a_i, b_i$ and $c_i$, respectively, such that:

1. $[I(a_i) \cup I(b_i) \cup I(c_i)] \cap C = A_i$ is open and closed;
2. $\text{diam } I(p) < \varepsilon$ for $p = a_i, b_i$ and $c_i$; and
3. if $T_j = \{a_j, b_j, c_j\}$ meets two of these intervals, then $A_i \supset T_j$.

Then there is a cellular use decomposition $G$ of $E^3$ such that $E^3/G$ contains a homeomorphic copy of $K$.

4. The nonexistence of a three-to-one mapping of $C$ onto a 2-simplex

If there is a three-to-one (continuous) mapping $f$ of $C$ onto a 2-simplex $\sigma^2$ such that $G^3_f$ is a null collection, then one can modify $f$ so that for each $y$ in a face (1-simplex) of $\sigma^2$, $f^{-1}(y)$ is at most two points. It is not difficult to see that this would imply the existence of a mapping $g$ onto the projective plane $P$ such that $g$ is at most three-to-one and $G^3_g$ is a null collection. We now state the following:

Theorem 5. There is no mapping $f$ of $C$ (the Cantor Space) onto a 2-simplex $\sigma^2$ such that $f$ is at most three-to-one and $G^3_f$ is a null collection.

Proof. If the theorem is false, then construct (as indicated above) an at most three-to-one mapping $g$ of $C$ onto the projective plane $P$ such that $G^3_g$ is a null collection. By Theorem 3, there is a closed mapping $\Psi$ of $E^3$ onto a metric space $(Y, d)$ where $Y$ contains a homeomorphic copy $Q$ of $P$.

Let $X = \Psi^{-1}(Q)$, a compactum in $E^3$, and let $\Theta = \Psi|_X$. Using the integers $Z$ for coefficients, the second cohomology group $H^2(Q)$ of $Q$ is isomorphic to $Z_2$ (integers mod 2). By a Vietoris-Begle theorem [8], $\Theta$ induces an isomorphism on Čech cohomology. Hence, $\tilde{H}(X)$ is isomorphic to $Z_2$. By Alexander duality [8], $\tilde{H}(X) \simeq \tilde{H}_0(E^3 - X) \simeq Z_2$. However, this is impossible. Hence, the theorem is proved.

We are indebted to Ross Geoghegan for suggesting this proof.

For similar reasons there is no admissible mapping $f$ of $C$ onto the
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projective plane $P$.

**Question.** For what compacta $K$ are there admissible mappings $f$ from the Cantor Space $C$ onto $K$?

### 5. Certain $k$-to-1 mappings, $k \geq 3$, raise dimension by at most $k - 2$.

A well-known theorem of Hurewicz [6, p.91] states that if $f$ is a closed mapping of a separable metric space $(X, d)$ onto a metric space $(Y, \rho)$ and $\dim X - \dim Y = k > 0$, then there is a point $y \in Y$ such that $\dim f^{-1}(y) \geq k$. A kind of dual of this theorem states that if $\dim Y - \dim X = k > 0$, then there is at least one point $y \in Y$ such that $|f^{-1}(y)| \geq k + 1$.

It is also well-known that if $Y$ is a $k$-dimensional compactum, then there is a mapping $f$ of the Cantor Space $C$ onto $Y$ such that for each $y \in Y$, $|f^{-1}(y)| \leq k + 1$. The question of the dimensionality of the image $Y$ of $C$ under a continuous mapping $f$ such that for each $y \in Y$, $|f^{-1}(y)| \leq k$ seems not to have been answered (and, maybe, not asked).

**Notation.** Suppose that $f$ is a finite-to-one mapping of $X$ onto $Y$. Let $N(x, f) = |f^{-1}(x)|$ for $x \in X$. Let $N(f) = \sup\{N(x, f) | x \in X\}$. Let $H_i(f) = \{y | y \in Y \text{ and } N(x, f) \geq i \text{ for } x \in f^{-1}(y)\}$.

In a letter, John J. Walsh gave a short easy proof that if $f$ is a continuous mapping of $C$ onto a compactum $Y$ such that $N(f) = 3$, $H_3(f) = \{y_1, y_2, \ldots\}$ is countable and $\{f^{-1}(y_i)\}$ is a null sequence, then $\dim Y \leq 1$. We were able to find a proof of a much more general result which is given below. It is an interesting result which seems to have been overlooked by dimension theorists.

**Theorem 6.** Suppose that each of $(X, d)$ and $(Y, \rho)$ is a separable metric space and that $f$ is a continuous mapping of $X$ onto $Y$ such that $f^{-1}f(x)$ is finite for each $x \in X$. If $k \geq 3$, $H_k(f) = \{y_1, y_2, \ldots\}$ and $\{f^{-1}(y_i)\}$ is a null sequence, then $\dim Y \leq \dim X + k - 2$.

**Proof.** The sequence $\{f^{-1}(y_i)\}$ is a null sequence. Thus, there are closed neighborhoods $V_i$ of $y_i$ such that $f^{-1}(V_i) = U_i^1 \cup U_i^2 \cup \cdots \cup U_i^n$, where each $U_i^s$ is closed, $U_i^s \cap U_i^t = \emptyset$ if $s \neq t$, each $U_i^s$ contains exactly one point of $f^{-1}(y_i)$, and, if $y \neq y_i$ and $f^{-1}(y)$ meets at least two of the sets $U_i^s$, then $|f^{-1}(y)| \leq k - 1$. Note that the set $K_i = \{y | f^{-1}(y) \text{ meets at least two of the sets } U_i^s\}$ is closed. Let $L_i = \{y | f^{-1}(y) > 1 \text{ and } f^{-1}(y) \text{ is a subset of some } U_i^s\}$. Now, for each natural number $i$ and each natural number $n,$
let \( L_i(n) = \{ y | y \in L_i \text{ and } \text{diam } f^{-1}(y) \geq 1/n \} \). Thus, \( L_i(n) \) is closed for each \( i \) and \( n \). The set \( W_i(n) = \text{int}(V_i - L_i(n)) \) is an open set.

Since \( g = f|_{f^{-1}(K_i)} \) is a closed mapping and \( N(x, g) \leq k - 1 \), except for \( x \) such that \( f(x) = y_i \), it follows that \( \dim K_i \leq \dim X + k - 2 \). (That \( k \geq 3 \) is needed here.) For \( y \in W_i(n) - y_i, |f^{-1}(y)| \leq k - 1 \). Thus, \( W = \bigcup_{i=1}^{n} \left( \bigcup_{n=1}^{\infty} W_i(n) \right) \) is open, and \( \dim W \leq \dim X + k - 2 \), since \( f|_{f^{-1}(W)} \) is closed and \( W \supset H_k(f) \). If \( y \notin W \), then \( |f^{-1}(y)| \leq k - 1 \). Thus, \( f|(X - f^{-1}(W)) \) is an at most \((k - 1)\)-to-one closed mapping on a closed set. This implies that \( \dim(Y - W) \leq \dim X + k - 2 \). Since \( W \) is an \( F_\sigma \)-set and \( Y - W \) is closed, \( \dim Y \leq \dim X + k - 2 \). The theorem is proved.

Remarks. There is an obvious two-to-one mapping \( f \) of the Cantor Space onto the closed interval \([0, 1]\) such that \( H_2(f) \) is a countable set \( \{ y_1, y_2, \ldots \} \) and \( \{ f^{-1}(y_i) \} \) is a null sequence. Thus, the assumption that \( k \geq 3 \) is necessary. However, there is no two-to-one mapping of the Cantor Space onto the plane one-dimensional Sierpinski curve such that \( H_2(f) \) is countable.

It would be interesting to classify those one-dimensional images of the Cantor Space under two-to-one mappings \( f \) such that \( H_2(f) \) is a countable set \( \{ y_1, y_2, \ldots \} \) and \( \{ f^{-1}(y_i) \} \) is a null sequence. More generally, we may attempt to classify those spaces \( Y \) such that there is a (continuous) finite-to-one mapping \( f \) of \( X \) onto \( Y \) such that \( k \geq 3 \), \( H_k(f) = \{ y_1, y_2, \ldots \} \) is countable and \( \{ f^{-1}(y_i) \} \) is a null sequence. In some sense, these spaces \( Y \) which have \( \dim Y \leq \dim X + k - 2 = m \) are "thinner" than some spaces of dimension \( m \).

References


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