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GENUS POLYNOMIALS OF DIPOLES

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The genus distribution of a graph G is the sequence $g_0, g_1, \cdots$, where g_m is the number of different 2-cell embeddings of G into the closed orientable surface of genus m. J. L. Gross *et al.* computed the *genus distribution* for a bouquet of circles, and asked for the genus distribution for other interesting graphs. In this paper, we compute the *genus distribution* for dipoles; that is, the multigraph having 2 vertices and multiple edges joining them.

1. Introduction

Let G be a finite connected graph allowing loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$, and let $|X|$ denote the cardinality of a set X. Convert G to a digraph by replacing each edge of G with a pair of oppositely directed edges. By $N(v)$, we denote the set of directed edges starting at $v \in V(G)$. An embedding of G into a closed surface S is a mapping $i : G \to S$ of G into S that corestricts to a homeomorphism $i : G \to i(G)$. If every component of $S - i(G)$, called a *region*, is an open disk, then the embedding $i : G \to S$ is called a 2-cell embedding. Two embeddings $i : G \to S$ and $j : G \to S$ of a graph G into an oriented surface S are *equivalent* if there is an orientation-preserving homeomorphism $h : S \to S$ such that $hi = j$. A rotation scheme ρ for a graph G is a map which assigns a cyclic permutation $\rho(v)$ of $N(v)$ to each $v \in V(G)$. It is well known (see, for example, [3] or Chapter 3 of [6]) that every rotation scheme ρ for a graph G determines a 2-cell embedding of G into an oriented surface S, and every 2-cell embedding of G is determined by such a scheme; in fact, there is a one-to-one correspondence between the set of rotation

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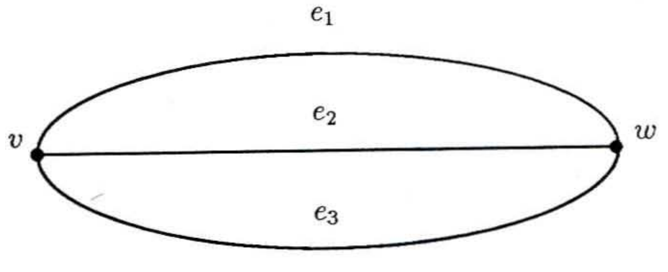
schemes for G and the equivalence classes of 2-cell embeddings of G into an oriented surface S. Throughout this paper, all surfaces are closed and orientable, all embeddings of graphs into surfaces are 2-cell embeddings, and the number of embeddings means the number of equivalence classes of embeddings.

The *genus distribution* of a graph G is defined to be the sequence $\{g_m\}$ such that g_m is the number of embeddings of the graph G into the surface of genus m. The *genus polynomial* of the graph G is defined by

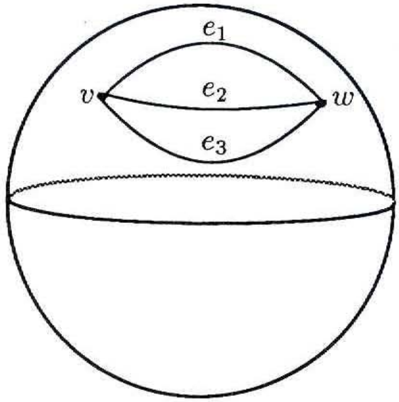
$$g[G](x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_N x^N,$$

where N is the highest genus in which G has a 2-cell embedding. Since knowing the genus polynomial implies knowing the genus distribution, we aim to compute in this paper the genus polynomial of the dipole D_n, which consists of two vertices joined by n edges.

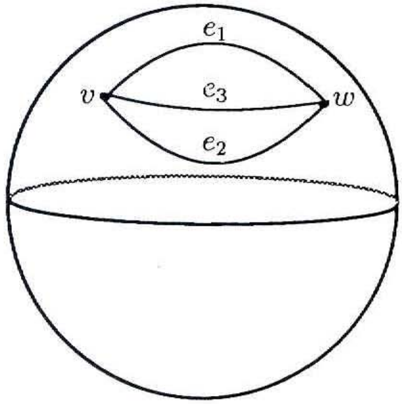
To get acquainted with the problem, we give an example of the embeddings of a particular dipole. Let $G = D_3$, which can be drawn as follows:



For each edge e_i, let e_i^+ denote the edge e_i with the direction from v to w and e_i^- the inverse edge of e_i^+ for $i = 1, 2, 3$. We can easily construct two nonequivalent embeddings of D_3 into the sphere S^2:



$i : D_3 \to S^2$



$j : D_3 \to S^2$

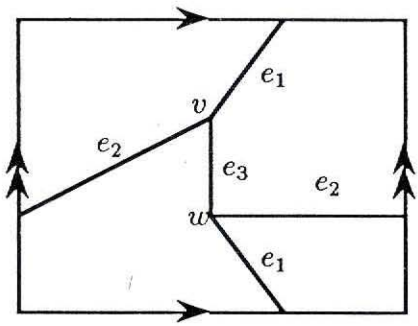
These two embeddings i and j are determined by the rotation schemes

$$\rho_i(v) = (e_1^+\ e_3^+\ e_2^+), \quad \rho_i(w) = (e_1^-\ e_2^-\ e_3^-),$$

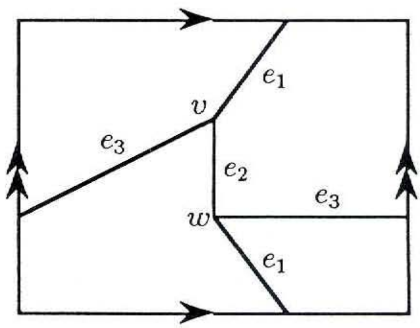
and

$$\rho_j(v) = (e_1^+\ e_2^+\ e_3^+), \quad \rho_j(w) = (e_1^-\ e_3^-\ e_2^-),$$

respectively. Note that these two embeddings are actually planar. The following figures show two different embeddings of D_3 into the torus T.



$k : D_3 \to T$



$\ell : D_3 \to T$

Their corresponding rotation schemes are

$$\rho_k(v) = (e_1^+\ e_2^+\ e_3^+), \quad \rho_k(w) = (e_1^-\ e_2^-\ e_3^-),$$

and

$$\rho_\ell(v) = (e_1^+\ e_3^+\ e_2^+), \quad \rho_\ell(w) = (e_1^-\ e_3^-\ e_2^-).$$

In fact, the only possible embeddings of D_3 into any surface are the above four, as will be shown in Section 3.

2. Rotation schemes for the dipole D_n

For every rotation scheme ρ for G, let $r(G,\rho)$ and $g(G,\rho)$ denote the number of regions and the genus of the surface in the embedding of G determined by ρ, respectively. Then we have $g(G,\rho) = \frac{1}{2}(2 - |V(G)| + |E(G)| - r(G,\rho))$ by the invariance of the Euler characteristic. Thus, computing $g(G,\rho)$ is equivalent to computing $r(G,\rho)$ for a given graph G and a rotation scheme ρ for G. Moreover,

$$g(D_n,\rho) = \frac{1}{2}(2-2+n-r(D_n,\rho)) = \frac{1}{2}(n - r(D_n,\rho)).$$

Hence, we get

Lemma 1. *For any rotation scheme ρ for D_n,*

$$r(D_n,\rho) = n - 2g(D_n,\rho).$$

In particular, the numbers n and $r(D_n,\rho)$ are either both even or both odd.

Let Σ_n be the set of all cyclic permutations in the symmetric group S_n. For any $\sigma \in S_n$, let $j(\sigma) = (j_1, \ldots, j_n)$ be the cycle type of σ, *i.e.*, j_k is the number of k-cycles occurring in the presentation of σ as the product of disjoint cycles.

Let $e_1, \ldots, e_n$ be the edges of D_n and v, w the vertices of D_n. Let e_i^+ be the edge with the direction from v to w and e_i^- the inverse edge of e_i^+ for each $i = 1, \ldots, n$. Let ρ be a rotation scheme for D_n viewed as a permutation on the directed edge sets of D_n, and let β denote the permutation $(e_1^+\ e_1^-)\cdots(e_n^+\ e_n^-)$ of the directed edges of D_n. Then, the regions of the embedding associated with the rotation scheme ρ are given by the cycles of the permutation $\rho(v)\rho(w)\beta$. Moreover, the number $r(D_n,\rho)$ of regions of the embedding equals the number of disjoint cycles of $\rho(v)\rho(w)\beta$ (*cf.* Theorem 2.1 in [5]).

For each rotation scheme ρ for D_n, by writing $\rho(v) = (e_1^+\ e_{k_2}^+ \cdots\ e_{k_n}^+)$ and $\rho(w) = (e_1^-\ e_{\ell_2}^- \cdots\ e_{\ell_n}^-)$ we can define $\bar\rho : V(D_n) \to \Sigma_n$ by $\bar\rho(v) = (1\ k_2\ \cdots\ k_n)$ and $\bar\rho(w) = (1\ \ell_2\ \cdots\ \ell_n)$. To simply further computations, we identify a rotation scheme ρ for D_n with the map $\bar\rho : V(D_n) \to \Sigma_n$.

Lemma 2. *For each rotation scheme ρ for D_n,*

$$r(D_n, \rho) = \sum_{k=1}^{n} j_k,$$

where $(j_1, \ldots, j_n)$ is the cycle type of $\bar{\rho}(v)\bar{\rho}(w)$.

Proof. Let $(e_i^+ \cdots)$ be a cycle occurring in the presentation of $\rho(v)\rho(w)\beta$ as the product of disjoint cycles. Then the cycle $(e_i^+ \cdots)$ is of the form

$$(e_i^+ \quad \rho(w)(e_i^-) \quad \rho(v)\beta(\rho(w)(e_i^-)) \quad \rho(w)\beta(\rho(v)\beta(\rho(w)(e_i^-))) \qquad \cdots),$$

where every odd term is an edge from v to w and every even term is an edge from w to v. This cycle is completely determined by the subcycle consisting of their odd terms, and this subcycle corresponds to the cycle

$$(i \quad \bar{\rho}(v)\bar{\rho}(w)(i) \quad (\bar{\rho}(v)\bar{\rho}(w))^2(i) \qquad \cdots)$$

in $\bar{\rho}(v)\bar{\rho}(w)$. This correspondence is clearly one-to-one from the set of disjoint cycles of $\rho(v)\rho(w)\beta$ onto the set of disjoint cycles of $\bar{\rho}(v)\bar{\rho}(w)$. In particular, the number of disjoint cycles of $\rho(v)\rho(w)\beta$ is equal to that of $\bar{\rho}(v)\bar{\rho}(w)$. This completes the proof.

Remark. A region of the embedding associated with a rotation scheme ρ is given by a cycle of the permutation $\rho(v)\rho(w)\beta$. The number of sides of the region equals the length of the corresponding cycle in $\rho(v)\rho(w)\beta$, which is two times the length of the corresponding cycle in $\bar{\rho}(v)\bar{\rho}(w)$, as shown in the proof of the above lemma.

Let $\bar{g}_m$ denote the number of embeddings of the graph D_n into the surface of genus m, or equivalently having $n - 2m$ regions, such that their corresponding permutation $\bar{\rho}$ satisfies $\bar{\rho}(v) = (1\,2\,\cdots\,n)$. Let

$$\bar{g}[D_n](x) = \bar{g}_0 + \bar{g}_1 x + \bar{g}_2 x^2 + \cdots.$$

Then by the symmetry of D_n we have the following theorem.

Theorem 1.

$$g[D_n](x) = (n-1)!\,\bar{g}[D_n](x).$$

We will compute $\bar{g}[D_n](x)$ in the following section by using D. M. Jackson's counting formula concerning the cycle structure of permutations ([7]).

3. Genus polynomials and Stirling numbers

Let σ denote the cycle $(1\,2\,\cdots\,n)$ throughout this section. In Section 2, we have seen that the number $\bar{g}_m$ of embeddings of D_n into the surface of genus m with $\bar{\rho}(v) = \sigma$ equals the number of $\bar{\rho} : V(D_n) \to \Sigma_n$ such that $\bar{\rho}(v) = \sigma$ and the permutation $\bar{\rho}(v)\bar{\rho}(w)$ has exactly $n - 2m$ disjoint cycles. Hence, to compute $\bar{g}[D_n](x)$, we need to count the number of $\tau \in \Sigma_n$ with the property that $\sigma\tau$ has exactly k cycles for each fixed number k. Jackson denoted this number by $e_k^{(n)}(1)$; we write it as $e(n,k)$. Note that this number $e(n,k)$ means the number of embeddings of D_n into the surface having exactly k regions such that their corresponding permutation $\bar{\rho}$ satisfies $\bar{\rho}(v) = \sigma$, that is, $e(n,k) = \bar{g}_{\frac{n-k}{2}}$.

The *Stirling numbers of the first kind* $s(n,k)$, (say the Stirling numbers simply), are defined as the coefficients of

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k.$$

Jackson computed the number $e(n,k)$ in terms of the Stirling numbers $s(n+1,k)$ as follows ([7], Theorem 5.4):

$$e(n,k) = \frac{1}{n+1}\sum_{\ell=0}^{n-k} n^{\ell}\binom{\ell+k+1}{k} s(n+1,\ell+k+1).$$

We summarize our discussions as the following theorem.

Theorem 2.

$$g[D_n](x) = (n-1)!\sum_{m=0}^{[\frac{n-1}{2}]} e(n,n-2m)x^m,$$

where

$$e(n,n-2m) = \frac{1}{n+1}\sum_{\ell=0}^{2m} n^{\ell}\binom{\ell+n-2m+1}{n-2m} s(n+1,\ell+n-2m+1).$$

To estimate the number $e(n,k)$, define

$$f(x) = x(x-1)(x-2)\cdots(x-n) \quad \text{and} \quad g(x) = f(n-x),$$

so that

$$f(x) = \sum_{h=0}^{n+1} s(n+1,h)x^h.$$

By taking the k-th derivative of $(-1)^{n+1}f(x) = g(x)$, we get

$$(-1)^{n+1}f^{(k)}(x) = g^{(k)}(x) = (-1)^k f^{(k)}(n-x)$$

and

$$(-1)^{n-k+1}f^{(k)}(0) = (-1)^k g^{(k)}(0) = f^{(k)}(n).$$

But, $f^{(k)}(0) = k!s(n+1,k)$. Hence, we have

Lemma 3. *For any k,*

$$f^{(k)}(n) = (-1)^{n-k+1}k!s(n+1,k).$$

Now, we state a formula for $e(n,k)$.

Theorem 3.

$$e(n,k) = \begin{cases} \frac{-2}{n(n+1)}s(n+1,k) & \textit{if } n-k \textit{ is even,} \\ 0 & \textit{if } n-k \textit{ is odd.} \end{cases}$$

Proof. From $f(x) = \sum_{h=0}^{n+1} s(n+1,h)x^h$, we have

$$\begin{aligned} f^{(k)}(x) &= \sum_{h=k}^{n+1} h(h-1)\cdots(h-k+1)s(n+1,h)x^{h-k} \\ &= k!s(n+1,k) + \sum_{h=k+1}^{n+1} h(h-1)\cdots(h-k+1)s(n+1,h)x^{h-k} \\ &= k!s(n+1,k) + \sum_{\ell=0}^{n-k}(\ell+2)\cdots(\ell+k+1)s(n+1,\ell+1+k)x^{\ell+1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{n}f^{(k)}(n) \\ &= \frac{k!}{n}s(n+1,k) + \sum_{\ell=0}^{n-k}(\ell+2)\cdots(\ell+k+1)s(n+1,\ell+1+k)n^{\ell} \\ &= \frac{k!}{n}s(n+1,k) + k!(n+1)\left(\frac{1}{n+1}\sum_{\ell=0}^{n-k}\binom{\ell+1+k}{k}s(n+1,\ell+1+k)n^{\ell}\right) \\ &= \frac{k!}{n}s(n+1,k) + k!(n+1)e(n,k). \end{aligned}$$

Hence,

$$e(n,k) = \frac{1}{k!n(n+1)}(f^{(k)}(n) - k!s(n+1,k)).$$

Now, Lemma 3 gives

$$e(n,k) = \frac{1}{n(n+1)}((-1)^{n-k+1}s(n+1,k) - s(n+1,k)).$$

This completes the proof.

By using Theorem 3, we can rewrite Theorem 2 as follows.

Theorem 4.

$$g[D_n](x) = \frac{-2(n-1)!}{n(n+1)} \sum_{m=0}^{[\frac{n-1}{2}]} s(n+1, n-2m)x^m.$$

Corollary 1.

(1) The maximum and minimum genera of D_n are $\left[\frac{n-1}{2}\right]$ and 0, respectively.

(2) There are exactly $(n-1)!$ planar embeddings of D_n.

(3) There are exactly $\frac{1}{24}(n+1)!(n-1)(n-2)$ toroidal embeddings of D_n.

(4) The number of embeddings of D_n having only one region is

$$(n-1)!e(n,1) = \begin{cases} \frac{2((n-1)!)^2}{n+1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

For example,

$$\begin{aligned}
g[D_2](x) &= 1. \\
g[D_3](x) &= 2(1+x). \\
g[D_4](x) &= 6(1+5x). \\
g[D_5](x) &= 24(1+15x+8x^2). \\
g[D_6](x) &= 120(1+35x+84x^2). \\
g[D_7](x) &= 720(1+70x+469x^2+180x^3). \\
g[D_8](x) &= 5040(1+126x+1869x^2+3044x^3). \\
g[D_9](x) &= 40320(1+210x+5985x^2+26060x^3+8064x^4).
\end{aligned}$$

Finally, we show that the genus distribution of the dipole D_n is strongly unimodal.

A non-negative sequence $\{a_n\}$ is said to be *unimodal* if there exists at least one integer M such that

$$a_{n-1} \leq a_n \quad \text{for all } n \leq M,$$

and

$$a_n \geq a_{n+1} \quad \text{for all } n \geq M.$$

A sequence $\{a_n\}$ is called *strongly unimodal* if its convolution with any unimodal sequence $\{b_n\}$ is unimodal. Keilson and Gerber [8] proved that $\{a_n\}$ is strongly unimodal if and only if

$$a_n^2 \geq a_{n+1}a_{n-1} \quad \text{for all } n.$$

For a fixed n, the sequence $e(n,n), e(n,n-2), \ldots, e(n, n-2[\frac{n-1}{2}])$ is finite. We want to show that

$$e(n,n-2k)^2 \geq e(n,n-2k-2)e(n,n-2k+2)$$

for $1 \leq k \leq [\frac{n-1}{2}] - 1$. Since $e(n,n-2k) = -\frac{2}{n(n+1)}s(n+1,n-2k)$, it is sufficient to show that

$$s(n+1,n-2k)^2 \geq |s(n+1,n-2k-2)|\,|s(n+1,n-2k+2)|.$$

But, it is known that

$$s(n,k)^2 \geq |s(n,k-1)|\,|s(n,k+1)|\,\frac{k(n-k+1)}{(k-1)(n-k)}$$

for $1 < k < n$ (See [2] pp. 270–271). This implies that

$$s(n,k)^2 \geq |s(n,k-1)|\,|s(n,k+1)| \quad \text{for } 1 < k < n.$$

Thus,

$$s(n+1,n-2k)^2 \geq |s(n+1,n-2k-1)|\,|s(n+1,n-2k+1)|$$

and

$$\begin{aligned} s(n+1,n-2k)^4 &\geq s(n+1,n-2k-1)^2 s(n+1,n-2k+1)^2 \\ &\geq |s(n+1,n-2k-2)|\, s(n+1,n-2k)^2\, |s(n+1,n-2k+ \end{aligned}$$

Hence,

$$s(n+1,n-2k)^2 \geq |s(n+1,n-2k-2)||s(n+1,n-2k+2)|.$$

The above discussion gives the following theorem.

Theorem 5. *The genus distribution of the dipole D_n is strongly unimodal.*

4. Further remarks

Let G be a connected graph and let ρ be a rotation scheme for G. Then ρ induces a multivariate monomial in the following manner. For each positive integer j, the exponent of the variable z_j equals the number of j-sided regions in the embedding. The sum of these monomials, taken over all embeddings, is called the *embedding polynomial* for the graph G. Recall that a rotation scheme ρ for D_n is identified with $\bar{\rho} : V(D_n) \to \Sigma_n$. Let $\bar{\rho}(v) = \sigma$ and let $\bar{\rho}(w) = \tau$. Then the contribution of ρ in the embedding polynomial $\imath[D_n][z_j]$ of D_n is the monomial $\prod_{k=1}^{n} z_{2k}^{j_k}$, where $j(\sigma\tau) = (j_1, \ldots, j_n)$. Let $\bar{\imath}[D_n](z_j)$ denote the polynomial corresponding to the set $\{(\sigma,\tau) \mid \sigma = (12\cdots n), \tau \in \Sigma_n\}$. Then we have the following theorem.

Theorem 6.

$$\imath[D_n](z_j) = (n-1)!\, \bar{\imath}[D_n](z_j).$$

For example, if $n = 4$ then the set $\{(\sigma,\tau) \mid \sigma = (1\ 2\ 3\ 4), \tau \in \Sigma_4\}$ has six elements.
If $\tau = (1\ 2\ 3\ 4)$, then $\sigma\tau = (1\ 3)(2\ 4)$.
If $\tau = (1\ 2\ 4\ 3)$, then $\sigma\tau = (1\ 3\ 2)(4)$.
If $\tau = (1\ 3\ 2\ 4)$, then $\sigma\tau = (1\ 4\ 2)(3)$.
If $\tau = (1\ 3\ 4\ 2)$, then $\sigma\tau = (1\ 4\ 3)(2)$.
If $\tau = (1\ 4\ 2\ 3)$, then $\sigma\tau = (1)(2\ 4\ 3)$.
If $\tau = (1\ 4\ 3\ 2)$, then $\sigma\tau = (1)(2)(3)(4)$.
Thus, $\bar{\imath}[D_4](z_j) = z_4^2 + 4z_2z_6 + z_2^4$, and $\imath[D_4](z_j) = 6z_4^2 + 24z_2z_6 + 6z_2^4$.

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independently obtained by Dr. R. G. Rieper in his Ph. D. thesis.

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