Optimal $k$ Value for the Profit Maximizing in the $k$ out of $n$ : open & close Systems.

Chung Hwan Oh*
Jong Chul Lee**

ABSTRACT

This Paper shows a special case of the optimization criterion is to make the maximum profit in the system reliability of the $k$ out of $n$ open & close structure. Especially, the number of the optimal $k$ is determined for the profit maximization in system reliability by deriving several properties of the optimal $k$ out of $n$ systems in one of four possible styles (closed & opened).

1. Introduction

The system consists of $n$ components (iid) that can be with a pre-specified frequency in one of the four possible styles. The components are subject to failures in each style. Thus, the four styles of component and system are

---

* The College of Suwon
** Osan Technical Engineering College
1) succeeding to close
2) failure to close
3) succeeding to open
4) failure to open

that is, the system is good to close if more than $k$ components close when closed and also is good to open if more than $k$ components of $n$ open when opened. In case of failure, the system is failure to close if fewer that $k$ components close when closed and is failure to open if fewer than $k$ components of $n$ open when trying to open [5]. A characterization of the optimal $k$ which maximizes the mean system-profit is obtained. This characterization is then applied to identify the situations under prediction, based on the parameters of the system, whether the optimal $k$ is smaller than or larger than $n/2$. The same characterization also predicts the effect on the optimal $k$ of a change in the costs of the four styles failure.

So, this paper shows to analyze and decide the optimal number of $k$ which maximizes the mean profit of the $k$ out of $n$ systems.

2. Notation

$I_1$: The probability of unit in succeeding to close
$I_2$: The probability of unit in failing to close
$I_4$: The probability of unit in succeeding to open
$I_5$: The probability of unit in failing to open

$n$: number of units in the system

$R_1(k)$: The probability of system in succeeding to close
$R_2(k)$: The probability of system in failing to close
$R_4(k)$: The probability of system in succeeding to open
$R_5(k)$: The probability of system in failing to open

$a$: The probability of the closed system

$1-a$: The probability of the opened system

$A'$: The same gained value from system success in close
$A''$: The same gained value from system failure in close
$A^*$: The same gained value from system success in open
$A^*$: The same gained value from system failure in open
3. The criterion and properties of the maximization for $k$

The maximization of:

$$Z(k) = R_{i}(k) - R_{r}(k) \{ (1-a)(A^i-A^r) / a(A^i-A^r) \}$$  \hspace{1cm} (1)

The optimization is straightforward to identify the results corresponding to this special case. A $k$ is optimal if and only if it satisfies (1), (2).

$$Z(k) - Z(k-1) \geq 0, \text{ and } Z(k) - Z(k+1) \geq 0 \text{ where } n > k > 0, \hspace{1cm} (2)$$

with at least one strict inequality.

For $k^* = 0$ or $n, k = 0$ is optimal value if and only if

$$Z(0) \geq Z(1) \hspace{1cm} (3)$$

and $k = 1$ is also optimal if and only if

$$Z(n) \geq Z(n-1) \hspace{1cm} (4)$$

From the equation (2) and (4), We can get the necessary and sufficient conditions for the determination of optimal $k$ are:

for $1 \leq k^* \leq n-1$

$$\left(\frac{I_i}{I_d}\right)^{n-k^*+1}(I : I_d)^{k^*-1} \leq (1-a)(A^i-A^r) / a(A^i-A^r) \leq \left(\frac{I_i}{I_d}\right)^{n-k^*}(I : I_d)^{k^*+1}$$ \hspace{1cm} (5) 

for $k^* = 0,$

$$\left(\frac{I_i}{I_d}\right)^* \geq (1-a)(A^i-A^r) / a(A^i-A^r)$$ \hspace{1cm} (6)

and for $k^* = n,$

$$\left(\frac{I_i}{I_d}\right)(I_i/I_d)^{-1} \leq (1-a)(A^i-A^r) / a(A^i-A^r)$$ \hspace{1cm} (7)

One of the uses of equation (5), (6), (7), is that it can help predict certain properties of the magnitude of $k^*$, based directly on the values of the parameters $(1-a)(A^i-A^r) / a(A^i-A^r)$. 

-146-
In particular, equation (5), (6), (7), delineates sufficient conditions under which \( k^* \) is:

\[
k^* < n/2 + 1; \text{ if } (1-a)(A^3-A^4)/a(A^1-A^2) < 1, \quad \text{and } I_1 \leq I_2 \tag{8}
\]

\[
k^* < n/2; \text{ if } (1-a)(A^3-A^4)/a(A^1-A^2) > 1, \quad \text{and } I_1 \geq I_2 \tag{9}
\]

\[
k^* = \lfloor n+1 \rfloor, \text{ for even } n; \text{ if } I_3 = I_4 \quad \text{and} \quad (1-a)(A^3-A^4)/a(A^1-A^2) = 1, \tag{10}
\]

\[
k^* = \lfloor n+1 \rfloor, \text{ for odd } n; k^* = n/2 \text{ or } n/2 + 1 \tag{11}
\]

(Proof of the Eq (8), (9), (10), (11))

We can rewrite Eq (5) as:

\[
(n-k^* + 1) \log \frac{I_1I_2}{I_3I_4} + (2k^*-n-2) \log \left( \frac{I_1}{I_3} \right) \\
\leq \log \left[ (1-a)(A^3-A^4)/a(A^1-A^2) \right] \\
\leq (n-k^*) \log \left( \frac{I_1I_2}{I_3I_4} \right) + (2k^*-n) \log \left( \frac{I_1}{I_3} \right) \tag{12}
\]

The right hand side of Eq (12) is nonnegative because \( \frac{I_1I_2}{I_3I_4} \geq 1, \ n-k^* + 1 > 0, \ \frac{I_1}{I_3} > 1, \ \text{and} \ (2k^*-n-2) \geq 0 \)

On the other hand, \( \log \left[ (1-a)(A^3-A^4)/a(A^1-A^2) \right] < 0. \)

Thus Eq (12) is contradicted.

Considering the case where \( \left\{ (1-a)(A^3-A^4)/a(A^1-A^2) \right\} = 1, \ \text{and} \ I_3 = I_4 \)

then \( \frac{I_1I_3}{I_3I_4} = 1 \ \text{from} \ \frac{I_1I_2}{I_3I_4} \geq 1, \ \text{if} \ I_3 \leq I_4. \)

Thus Eq (12) becomes
Optimal $k$ Value for the Profit Maximizing in the $k$ out of $n$ : open & close Systems  

\[ \log \frac{I_1}{I_3} \geq 0 \geq \frac{(2k^* - n) - 2}{2k^* - n - 2} \log \frac{I_1}{I_3} \]  

(13)

Now since $\log \frac{I_1}{I_3} > 0$, it follows from Eq (13) that $k^* = (n+1)/2$ if $n$ is odd. If $n$ is even, then Eq (13) is satisfied for either $k^* = n/2$, or $k^* = (n/2) + 1$

Theorem 1.

An interior $k^*$ is nondecreasing in $(1-a)(A^2-A^1)/a(A^1-A^2)$. If the increase in $(1-a)(A^3-A^1)/a(A^1-A^2)$ is sufficiently large, then an interior $k^*$ must increase.

(Proof)

From the Eq (5), We can rewrite as

\[ \left( \frac{(I_1 I_3)}{(I_1 I_3)} \right)^{k-1} \leq \left( \frac{(1-a)(A^1-A^1)}{a(A^1-A^2)} \right)^{k-1} \left( \frac{I_1 I_3}{I_1 I_3} \right)^{k-1} \]

where $(I_1 I_3) / (I_1 I_3) > 1$ because $I_1 > I_3$. Now let’s suppose $V = (1-a)(A^1-A^1)/a(A^1-A^2)$ is changed $\overline{V}$ such that $\overline{V} > V$. If the corresponding optimal $k$ is denoted $\overline{k}^*$, then

\[ \left( \frac{(I_1 I_3)}{(I_1 I_3)} \right)^{\overline{k}^*-1} \leq \frac{V}{(I_1 I_3)} \frac{(I_1 I_3)}{(I_1 I_3)} \leq \left( \frac{(I_1 I_3)}{(I_1 I_3)} \right)^{k} \]

(15)

Because of $\overline{V} > V$, the second part of the inequality (14) yields: $V / (I_1 I_3) > \left( \frac{(I_1 I_3)}{(I_1 I_3)} \right)^{\overline{k}^*-1}$. The preceding equation implies that the first part of the inequality (15) will be contradicted if $\overline{k}^* \leq \overline{k}^*-1$. Therefore $\overline{k}^* > k^*$ and if $\overline{V}$ is sufficiently larger than $V$, then it is obvious that Eq (15) will be satisfied only if $\overline{k}^* > k^*$.

4. Evaluation at $k^*$ by the result of the derivation for $Z(k)$

This part shows some effects concerning how an interior $k^*$ is altered due to a change in the number of components $n$, or due to a change in the parameters $I_1$ and $I_4$.

Our main project here is to derive qualitative effects and results with respect to the direction of change in optimal $k$ due to a change in parameters.

The derivative of $Z(k)$ with respect to $k$ is:

\[ -148 - \]
\[ Z(k) = R_{ik}(k) - R_{ik}(k)V \quad \text{(where} \quad V = (1-a)(A^i - A^i)/a(A^i - A^i)) \]

\[
R_{ik}(k) = \partial R_{ik}(k)/\partial k
\]

\[
R_{ik}(k) = \partial R_{ik}(k)/\partial k
\]

(16)

Thus, the interior extreme points of \( Z(k) \) are those which satisfy:

\[
Z_i(k) = 0
\]

It's very easy to show that \( Z_i(k) \) is strictly concave in \( k \) at any \( k \) which satisfies \( Z_i(k) = 0 \). It follows therefore that, for an interior \( k^* \), and \( Z(k) = 0 \) represents the necessary and sufficient condition for optimality (since \( \partial Z_i(k)/\partial k < 0 \))

Now, using the derivation for the evaluation at \( k^* \), we obtain the following expression for \( dk^*/dn \) that is:

\[
\frac{dk^*}{dn} = \frac{(k^*)^2 I_i I_i + \left(n^2 - (k^*)^2\right) I_i I_i}{\left(k^* I_i I_i + (n-k^*) I_i I_i\right)2n}
\]

(17)

(Proof)

\( Z_i \) in Eq (16) can be rewritten as \( Z_i = R_{ik}[(R_{ik}/R_{ik})-V] \).

(Where \( V = (1-a)(A^i - A^i)/a(A^i - A^i) \). Thus, the derivative \( dZ_i/dk \), evaluated at \( Z_i = 0 \), is:

\[
\frac{\partial Y_i}{\partial k} = R_{ik} \frac{\partial (R_{ik}/R_{ik})}{\partial k}
\]

\[
= [k^* I_i I_i + (n-k^*) I_i I_i] R_{ik}(I_i - I_i) / (n I_i I_i) < 0
\]

(18)

Therefore,

\[
\frac{\partial Z_i}{\partial n} = -R_{ik}(I_i - I_i) [(k^*)^2(1-I_i - I_i) + n^2 I_i I_i] / (2n I_i I_i I_i)
\]

(19)

and hence,

\[
\frac{dk^*}{dn} = [(k^*)^2 I_i I_i + \left(n^2 - (k^*)^2\right) I_i I_i] / \{2n(k^* I_i I_i + (n-k^*) I_i I_i)\}
\]

(20)

Eq (17) is same as Eq (20). So, the proof is concluded like as above developments.
Eq (20) yields several qualitative conclusions concerning the local change in an interior \( k^* \) as \( n \) changes.

\[
\frac{dk^*}{dn} \text{ is between 0 and 1, and if } I_3 \geq I_2 \text{ then } \frac{dk^*}{dn} \geq 1/2.
\]

The final result, the following theorem 2 ascertains the direction of local change in \( k^* \) from a change in the probabilities of unit failure, when these probabilities are the same in the four styles structure.

**Theorem 2.**

\[
\frac{dk^*}{dI_i} \geq 0, \text{ if } V \leq 1, \text{ and } I_3 = I_2 \tag{21}
\]

**Proof**

Like as the method and development of Eq (19)

\[
\frac{\partial Z_i}{I_i} = -R_{ik}(n-2k^*)\left(I_1^2 - I_1^2\right)/(2I_i^2 I_1^2) \tag{22}
\]

Using the \( \partial Z / \partial K < 0 \) and Eq (22), it follows that

\[
\frac{dk^*}{dI_i} \geq 0, \text{ if } \frac{k^*}{n} \leq 1/2, \tag{23}
\]

(when \( \frac{k^*}{n} \leq 1/2, \text{ if } V \leq 1, \text{ and } I_3 = I_2 \))

Thus Eq (21) in combination with Eq (22) implies that \( k^*/n \) becomes closer to half as the probabilities of unit failure become smaller.

**5. Concluded Explanation**

To test the fitability of these above results, we undertook numerical simulations for a range of parameters of exact change in \( K^* \) due to changes in parameters. The changes in \( K^* \) are computed as follows. For each combination of parameters, the integer value of \( K^* \) is
first calculated directly from \( Eq \ (5) \ (6) \ (7) \). One of the parameters is then altered, and the new integer value of \( K^* \) is similarly calculated.

The set of simulations was for each of the following \( 540 \ (5 \times 9 \times 3 \times 4 = 540) \) combinations of parameters: \( n = (25, 50, 75, 100, 125) \)

\[
I_1 = (0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9)
\]

\[
I_2 = (I_1 - 0.05, I_1 - 0.1, I_1 - 0.2)
\]

\[
V = (0.05, 0.1, 0.75, 1.5)
\]

In each case, the value of \( n \) was increased by 20, and the value of \( I_1 \) was increased by 0.05 and the resulting increase in the integer value of \( k^* \) was computed. For all cases for which the pre- and post-change \( k^* \) had an interior value, the change in \( k^* \) was non-negative that is, if \( I_1 \geq I_2 \), then \( 1 \leq k^* \leq 2 \), and if \( I_1 \leq I_2 \) then \( k^* \leq 1 \). And the change in \( k^* \) was non-negative for \( V = (1-a)(A^1-A^2) \) and nonpositive for \( V = (1-a)(A^1-A^2) \).

The objective of this paper was to analyze the \( K \) which maximizes the mean system profit. We show how one can predict, based on the parameters of the system, if the \( k \) is large of smaller than \( n/2 \). Also the positions of change in the \( k \) resulting from changes in system parameters are ascertained.

**REFERENCES**


