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CERTAIN ONE RELATOR CONJUGACY SEPARABLE GROUPS

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1. Introduction

A group G is said to be conjugacy separable (c.s.) if, whenever x and y are elements of G which are not conjugate, there is a finite homomorphic image of G in which the images of x, y are not conjugate. This concept is important to solve the conjugacy problem for finitely presented groups, since Mostowski [13] solved the conjugacy problem for finitely presented c.s. groups. For example, finitely generated (f.g.) nilpotent groups [2], free groups [14], polycyclic-by-finite groups [7], and free-by-finite groups [5] are c.s. On the other hand, Wehrfritz [15, 16] gave some soluble groups which are not c.s. Then Miller [12] constructed a generalized free product (g.f.p.) of free groups which is not c.s. However, Dyer [4] showed that the g.f.p. of free groups (or f.g. nilpotent groups) amalgamating a cyclic subgroup is c.s. In [1], Allenby and Tang showed that the groups $\langle a, b : (a^{-1}b^l ab^m)^s\rangle$ is c.s. for $s > 1$. Then Fine and Rosenberger [6] proved that all Fuchsian groups are c.s.

Recently, Kim and McCarron [10] found a condition for the g.f.p. of groups to be residually p-finite. Using this, they [11] characterized all residually p-finite groups of the form $\langle a, b : a^{-\alpha}b^{\beta}a^{\alpha}b^{\lambda}\rangle$. Thus, we are confronted with the task of classifying those one relator groups that are c.s. In this paper, we give an elementary proof that the one relator group $\langle a, b : a^{-1}ba = b^{\delta}\rangle$ is c.s. for any δ (Theorem 2.3). Using this, we give examples of c.s. or not c.s. g.f.p. of groups amalgamating a cyclic subgroup (Example 2.4, 2.5), which are having the solvable conjugacy problem. The existence of such groups was pointed out by Dyer [4].

We shall use the following terminology and result:

$x \sim_G y$ (simply $x \sim y$) means that there exists $g \in G$ such that $x = g^{-1}yg$, and we use $x \not\sim_G y$ (simply $x \not\sim y$) if there is no $g \in G$ such that $x = g^{-1}yg$. We use $\langle X\rangle^G$ to denote the normal closure of X in G. $N \lhd_f G$ denotes N is a normal subgroup of finite index in G. If $\overline{G}$ is a homomorphic image of G, then we use $\overline{x}$ to denote the image of $x \in G$ in $\overline{G}$. Finally, (n,m) denotes the greatest common divisor of n and m.

LEMMA 1.1. [11] *Let* $G = \langle a, b : a^{-1}ba = b^{\delta}\rangle$ *and let* $b_{\ell} = a^{\ell}ba^{-\ell}$ *for* $\ell \geq 0$. *If* $(\delta, k) = 1$ *then* $\langle b\rangle^G/\langle b_{\ell}^k\rangle^G$ *is a cyclic group of order* k *for any* $\ell \geq 0$. *Thus* $G/\langle b_{\ell}^k\rangle^G$ *is finite-by-cyclic, hence it is c.s.* [5].

2. Results

First we consider the following result on integers.

LEMMA 2.1. *Let* m, n, δ *be integers such that* $m \neq 0 \neq n$ *and* $|\delta| \geq 2$. *If* δ *does not divide* n *and if* $m \neq \delta^i n$ *for any* $i \geq 0$, *then there exists an integer* r *such that* $m \not\equiv \delta^i n \pmod{q}$ *for all* $i \geq 0$, *where* $q = \delta^{6r} - 1$.

proof. Case 1. $\delta \geq 2$.
Subcase 1. Consider $m > 0$ *and* $n < 0$. Choose an integer $r_0 \geq 3$ such that $0 < m, -n < \delta^{r_0+1}$. For each $r \geq r_0$, we can write

$$m = m_r\delta^r + m_{r-1}\delta^{r-1} + \cdots + m_1\delta + m_0 \text{ and}$$
$$-n = n_r\delta^r + n_{r-1}\delta^{r-1} + \cdots + n_1\delta + n_0,$$

where $0 \leq m_i, n_i < \delta$. Then $n_0 \neq 0$. For $0 \leq i \leq 2r-1$, we have

$$\begin{aligned} 0 < m - \delta^i n &= n_r\delta^{r+i} + n_{r-1}\delta^{r+i-1} + \cdots + n_1\delta^{i+1} + n_0\delta^i + m \\ &\leq (\delta-1)\{\delta^{3r-1} + \delta^{3r-2} + \cdots + \delta^{2r-1} + \delta^r + \cdots + 1\} \\ &= \delta^{3r} - 1 - \delta^{r+1}(\delta^{r-2} - 1) < \delta^{3r} - 1. \end{aligned}$$

Thus, for $0 \leq i \leq 2r-1$, we have $0 < m - \delta^i n < \delta^{3r} - 1$. Now, we have

$$\begin{aligned} 0 < m - \delta^{2r+j}n \equiv n_r\delta^j + \cdots + n_{r-j}\delta^0 + n_{r-j-1}\delta^{3r-1} + \\ \cdots + n_0\delta^{2r+j} + m \pmod{q = \delta^{3r} - 1}, \end{aligned} \tag{2.1}$$

for $0 \leq j \leq r-1$. Then the right hand side of (2.1) is equal to

$$\begin{aligned} &n_{r-j-1}\delta^{3r-1} + \cdots + n_0\delta^{2r+j} + m + n_r\delta^j + \cdots + n_{r-j}\delta^0 \\ &\leq n_{r-j-1}\delta^{3r-1} + \cdots + n_0\delta^{2r+j} + (\delta^{r+1} - 1) + (\delta^{r+1} - 1) \\ &\leq (\delta-1)\delta^{3r-1} + (\delta-1)\delta^{3r-2} + \cdots + (\delta-1)\delta^{2r} + 2\delta^{r+1} - 2 \\ &= \delta^{3r} - \delta^{2r} + 2\delta^{r+1} - 2 < \delta^{3r} - 1. \end{aligned}$$

Thus, since $n_0 \neq 0$, $m - \delta^{2r+j} n \not\equiv 0 \pmod{q = \delta^{3r} - 1}$. Hence, $m \not\equiv \delta^i n$ $(\text{mod } \delta^{3r} - 1)$ for all $i \geq 0$, where r is any integer $\geq r_0$.

Subcase 2. Consider $m > 0$ *and* $n > 0$. As before, we choose $r_0 \geq 3$ such that $0 \leq m, n < \delta^{r_0+1}$. For each $r \geq r_0$, we write $m = m_r\delta^r + \cdots + m_1\delta + m_0$ and $n = n_r\delta^r + \cdots + n_1\delta + n_0$, where $0 \leq m_i, n_i < \delta$. For $0 \leq i \leq 2r-1$, we have $-\delta^i n < m - \delta^i n < m$. Note that $\delta^i n \leq \delta^{3r} - \delta^{2r-1} < \delta^{3r} - 1$. Hence $m > m - \delta^i n > -(\delta^{3r} - 1)$. So $m - \delta^i n \not\equiv 0 \pmod{q = \delta^{3r} - 1}$, for $0 \leq i \leq 2r-1$. For $0 \leq j \leq r-1$, we have

$$(2.2) \qquad \begin{aligned} m - \delta^{2r+j} n \equiv{} & -n_r\delta^j - \cdots - n_{r-j}\delta^0 - n_{r-j-1}\delta^{3r-1} - \\ & \cdots - n_0\delta^{2r+j} + m \pmod{q = \delta^{3r} - 1}. \end{aligned}$$

Then the right hand side of (2.2) is greater than

$$\begin{aligned} & -n_r\delta^j - \cdots - n_{r-j}\delta^0 - n_{r-j-1}\delta^{3r-1} - \cdots - n_0\delta^{2r+j} \\ & \geq -\{(\delta-1)\delta^{3r-1} + (\delta-1)\delta^{3r-2} + \cdots + (\delta-1)\delta^{2r} + (\delta-1)\} \\ & = -(\delta^{3r} - \delta^{2r} + \delta - 1) > -(\delta^{3r} - 1). \end{aligned}$$

Now, since $n_0 \neq 0$, $m - \delta^{2r+j} n \not\equiv 0 \pmod{q = \delta^{3r} - 1}$ for $0 \leq j \leq r-1$. Hence, $m \not\equiv \delta^i n \pmod{\delta^{3r} - 1}$ for all $i \geq 0$, where r is any integer $\geq r_0$.

Subcase 3. $m < 0$ *and* $n > 0$. Since $-m > 0$ and $-n < 0$, by Subcase 1, there exists an integer r_0 such that $-m \not\equiv \delta^i(-n) \pmod{q}$ for all $i \geq 0$, where $q = \delta^{3r} - 1$ for any $r \geq r_0$. Hence $m \not\equiv \delta^i n$ (mod q) for all $i \geq 0$, where $q = \delta^{3r} - 1$ for any $r \geq r_0$.

Subcase 4. Consider $m < 0$ *and* $n < 0$. This case can be handled by Subcase 2 above.

Case 2. $\delta \leq -2$.

Let $\delta = -\delta_1$, where $\delta_1 \geq 2$. Since $m \neq \delta^i n$ for any $i \geq 0$, we have $m \neq (\delta_1^2)^i n$ and $-\delta_1 m \neq (\delta_1^2)^i n$ for any $i \geq 0$. By Case 1, there exist integers r_1, r_2 such that, for any $i \geq 0$, $m \not\equiv (\delta_1^2)^i n \pmod{(\delta_1^2)^{3r} - 1}$ for any $r \geq r_1$ and $-\delta_1 m \not\equiv (\delta_1^2)^i n \pmod{(\delta_1^2)^{3r} - 1}$ for any $r \geq r_2$. It follows that, for any $i \geq 0$, $m \not\equiv \delta^i n \pmod{q}$, where $q = \delta^{6r_1r_2} - 1$.

The next lemma will be useful to prove our main result.

LEMMA 2.2. *$G = \langle a, b;\ a^{-1}ba = b^\delta \rangle$, where $|\delta| \geq 2$.*
(a) $b^n \sim_G b^m$ if, and only if, $n = \delta^i m$ or $m = \delta^i n$ for some $i \geq 0$.

(b) For $s>0$, $b^m a^s \sim_G b^n a^s$ if, and only if, $m \equiv \delta^i n$ (mod $|\delta^s - 1|$) for some $i \geq 0$.

proof. For (a), we note that $b^n \sim_G b^m$ iff $b^n = a^{-i} b_t^{-j} b^m b_t^j a^i$ for some i, j and some $t \geq 0$, since $\langle b \rangle^G = \langle b_0, b_1, \ldots ; b_i = b_{i+1}^{\delta} \rangle$ is locally cyclic, and since $G = \langle b \rangle^G \langle a \rangle$. Then $b^n \sim_G b^m$ iff $b^n = a^{-i} b^m a^i$ iff $b^n = b^{\delta^i m}$ or $b^m = b^{\delta^i n}$, for some $i \geq 0$. Since $|b| = \infty$, we have the result (a).

(b): ($\Longleftarrow$) Let $m = \delta^i n + \lambda(1-\delta^s)$. Then $a^i b^{\delta^s \lambda}(b^m a^s) b^{-\delta^s \lambda} a^{-i} = a^i b^{\delta^s \lambda} b^m b_s^{-\delta^s \lambda} a^s a^{-i} = a^i b^{\delta^s \lambda - \lambda + m} a^{-i} a^s = a^i b^{\delta^i n} a^{-i} a^s = b^n a^s$.

($\Longrightarrow$) Suppose $b^m a^s \sim_G b^n a^s$. Then $b^m a^s = a^{-t} b_k^{-\lambda}(b^n a^s) b_k^{\lambda} a^t = a^{-t} b_k^{-\lambda} b^n b_{k+s}^{\lambda} a^s a^t = a^{-t} b_{k+s}^{-\delta^s \lambda + \delta^{k+s} n + \lambda} a^s a^t$ for some $k \geq 0$ and some t, λ. Thus $b^m a^s = b_{k+s}^{\delta^t(\delta^{k+s} n + (1-\delta^s)\lambda)} a^s$ or $b^{\delta^t m} a^s = b_{k+s}^{\delta^{k+s} n + (1-\delta^s)\lambda} a^s$ for $t \geq 0$. Since $b^m = b_\ell^{\delta^\ell m}$ and $|b_\ell| = \infty$ for any $\ell \geq 0$, we have $\delta^{k+s} m = \delta^t(\delta^{k+s} n + (1-\delta^s)\lambda)$ or $\delta^{k+s+t} m = \delta^{k+s} n + (1-\delta^s)\lambda$ for some $t \geq 0$. In any case, using $\delta^s \equiv 1$ (mod $|\delta^s - 1|$), we have $m \equiv \delta^i n$ (mod $|\delta^s - 1|$) for some $i \geq 0$.

Now we are ready to prove the main result.

THEOREM 2.3. *The group $G = \langle a, b : a^{-1} b a = b^\delta \rangle$ is c.s. for any integer δ.*

proof. If $\delta = 0, 1$ then G is free abelian, hence it is clearly c.s. For $\delta = -1$, G has nontrivial center, hence it is also c.s. [4]. Thus it suffices to consider the case for $|\delta| \geq 2$. Let $x \not\sim_G y$. Since $G = \langle b \rangle^G \langle a \rangle$, and since $\langle b \rangle^G$ is locally cyclic, we may write $x = b_l^n a^s$ and $y = b_l^m a^t$ for some n, m, s, t and some $l \geq 0$. If $s \neq t$, then $x\pi \not\sim y\pi$, where $\pi : G \to G/\langle b \rangle^G$ is a natural homomorphism. Since $G\pi$ is c.s., we can find $\overline{N} \lhd_f G\pi$ such that $x\pi\overline{N} \not\sim y\pi\overline{N}$ in $G\pi/\overline{N}$. Let $N = \pi^{-1}(\overline{N})$. Then clearly $N \lhd_f G$ and $xN \not\sim yN$ in G/N. So, from now, we consider $s = t$. Moreover, since $b_l^n a^s \sim b^n a^s$ for any $l \geq 0$, we may assume $x = b^n a^s$ and $y = b^m a^s$, where δ does not divide n and m.

Case 1. $s = 0$. By Lemma 2.2, we have $n \neq \delta^i m$ and $m \neq \delta^i n$ for $i \geq 0$. Then, by Lemma 2.1, there exists an even integer r such that $n \not\equiv \delta^i m$ (mod $\delta^r - 1$) for any $i \geq 0$. Let $q = \delta^r - 1$, then $(\delta, q) = 1$. In $\overline{G} = G/\langle b^q \rangle^G$, we have $\overline{x} = \overline{b}^n \not\sim \overline{b}^m = \overline{y}$. Since $\overline{G}$ is c.s. by Lemma 1.1, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{x}\overline{N} \not\sim \overline{y}\overline{N}$ in $\overline{G}/\overline{N}$. Let N

be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $xN \not\sim yN$ in G/N as required.

Case 2. $s > 0$. Note that if $k = \lambda(1-\delta^s)+k'$, where $0 \le k' < |\delta^s - 1|$, then $b^{-\delta^s\lambda}(b^{k'}a^s)b^{\delta^s\lambda} = b^{-\delta^s\lambda}b^{k'}b_s^{\delta^s\lambda}a^s = b^{k'+\lambda(1-\delta^s)}a^s = b^k a^s$, hence $b^{k'}a^s \sim b^k a^s$. Thus we may assume that $0 \le n, m < |\delta^s - 1|$. Now, since $b^n a^s \not\sim b^m a^s$, by Lemma 2.2, we have $m \not\equiv \delta^i n \pmod{|\delta^s - 1|}$ for any $i \ge 0$. Let $q = |\delta^s - 1|$, then $(\delta, q) = 1$. In $\overline{G} = G/\langle b^q \rangle^G$, we have $\overline{x} \not\sim \overline{y}$, and $\overline{G}$ is c.s. by Lemma 1.1. Hence, as in Case 1, we can find $N \lhd_f G$ and $xN \not\sim yN$ in G/N.

Case 3. $s < 0$. Let $s = -s'$. Then $x \not\sim y$ iff $x^{-1} \not\sim y^{-1}$. Now $x^{-1} = a^{s'}b^{-n} \sim b^{-n}a^{s'}$ and $y^{-1} \sim b^{-m}a^{s'}$, where $s' > 0$. Thus this case can be handled as in Case 2. This completes the proof.

Miler [12] constructed a g.f.p. of free groups amalgamating a f.g. subgroup which has not the solvable conjugacy problem. Then Dyer [4] noted that there exists a g.f.p. which has the solvable conjugacy problem but is not c.s. The following example may be compared with this fact. We shall use $A *_H B$ to denote the g.f.p. of A and B amalgamating the subgroup H.

EXAMPLE 2.4. The group $\langle a, b : a^{-1}ba = b^2 \rangle *_{\langle b \rangle} \langle c, b : c^{-1}bc = c^2 \rangle$ is not residually finite [8], hence it is not c.s. But we can solve the conjugacy problem for this group by [3].

EXAMPLE 2.5. The group $\langle a, b : a^{-1}ba = b^\delta \rangle *_{\langle a \rangle} \langle a, c, : a^{-1}ca = c^\delta \rangle$ is c.s. by [9] for any δ.

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