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AN COMPLETION OF SPACE OF FUZZY RANDOM VARIABLES

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1. Introduction

Fuzzy random variables generalize random sets which is an extension of random variables and random vectors. Kwakernaak[5] introduced the notion of a fuzzy random variable as a function $F : \Omega \to \overline{\mathcal{F}}(R)$ subject to certain measurability conditions,where(Ω, Σ, P) is a probability space, and $\overline{\mathcal{F}}(R)$ denotes all piecewise continuous functions $u : R \to [0, 1]$. Puri and Ralescu[7] defined a fuzzy random variable by a function $X : \Omega \to \mathcal{F}_o(R^n)$ subject to certain measurability requirements, where $\mathcal{F}_o(R^n)$ denotes all functions $u : R^n \to [0, 1]$ such that $\{x \in R^n : u(x) \geq \alpha\}$ is nonempty and compact for each $0 < \alpha \leq 1$, and proved an completion of $\mathcal{F}_o(R^n)$ with respect to an appropriate metric. Stojakovic[9] defined the notion of a fuzzy random variable slightly different than that in [5] and [7], and proved that the space of integrably bounded fuzzy random variables is complete with repect to a new metric.

In this paper, we adopt the notoin of a fuzzy random variable in Puri and Ralescu[7], and the space of integrably bounded fuzzy random variables is complete with respect to the metric introduced in Stojakovic[9].

2. Preliminaries

Throughout this paper, let (Ω, Σ, P) be a probability space and Λ a real separable Banach space with norm $\|\ \ \|$. Let $\mathcal{K}(\Lambda)$ denotes the family of all nonempty, compact subsets of Λ and $\mathcal{K}_c(\Lambda)$ the family of all nonempty, compact, and convex subsets of Λ. A linear structure in $\mathcal{K}(\Lambda)$ is defined via the operations

$$A + B = \{a + b : a \in A, b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in \mathcal{K}(\Lambda), \lambda \in R$. However, note that $\mathcal{K}(\Lambda)$ is not a vector space since $A + (-A) \neq \{0\}$.

The topology in $\mathcal{K}(\Lambda)$ is introduced through the Hausdorff metric

$$H(A, B) = max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

We denote the Hausdorff semimetric by

$$h(A, B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$$

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It is well-known that $\mathcal{K}(\Lambda)$ is a complete and separable metric space, and that $\mathcal{K}_c(\Lambda)$ is a closed subspace.
Let $L(\Omega, \Sigma, P, \Lambda) = L$ denotes the Banach space of (equivalence classes of) measurable functions $f : \Omega \to \Lambda$ such that the norm $\|f\|_1 = \int_\Omega \|f(\omega)\|\, dP$ is finite.
A random set is defined as a Borel measurable function $F : \Omega \to \mathcal{K}(\Lambda)$, and a measurable function $f : \Omega \to \Lambda$ is called a measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. For a random set F, define the set $S_F = \{f \in L : f(\omega) \in F(\omega) a.e.\}$ then , S_F is a closed subset of L. If $F : \Omega \to \mathcal{K}(\Lambda)$ is a random set, the expectation of F is defined by $\int_\Omega F\, dP = \{\int_\Omega f\, dP : f \in S_F\}$ where $\int_\Omega f\, dP$ is the Bochner-integral. A random set $F : \Omega \to \mathcal{K}(\Lambda)$ is called integrably bounded if there exists integrable function $g : \Omega \to R$ such that $\sup_{x\in F(\omega)} \|x\| \leq g(\omega)$ for all $\omega \in \Omega$. Let $\mathcal{L}(\Omega, \Sigma, P, \Lambda) = \mathcal{L}$ denote the space of all integably bounded random sets, where $F, G \in \mathcal{L}$ are considered to be identical if $F(\omega) = G(\omega)$ a.s.. For $F, G \in \mathcal{L}$, we define

$$\Delta(F, G) = \int_\Omega H\{F(\omega), G(\omega)\}\, dP$$
$$\delta(F, G) = \int_\Omega h\{F(\omega), G(\omega)\}\, dP$$

Then Δ is a metric and δ is a semimertic on $\mathcal{L}$. If we define

$$\mathcal{L}_c(\Omega, \Sigma, P, \Lambda) = \mathcal{L}_c = \{F \in \mathcal{L} : F(\omega) \in \mathcal{K}_c(\Lambda) a.s.\}$$

then $\mathcal{L}$ is a complete metric space with respect to the metric Δ and $\mathcal{L}_c$ is a closed subspace of $\mathcal{L}$ [3].

3. Fuzzy random variables

A Fuzzy set in Λ is a function $u : \Lambda \to [0,1]$. Denote by $L_\alpha u = \{x \in \Lambda | u(x) \geq \alpha\}$ for $0 \leq \alpha \leq 1$, the α-level set of u. An extension of $\mathcal{K}(\Lambda)$ is obtained by defining the space $\mathcal{F}(\Lambda)$ of all fuzzy sets $u : \Lambda \to [0,1]$ with the properties

(a) u is upper semicontinuous

(b) supp u is compact

(c) $\{x \in \Lambda | u(x) = 1\} \neq \emptyset$

The space $\mathcal{F}_c(\Lambda)$ denotes the family of all fuzzy sets in $\mathcal{F}(\Lambda)$ which are also fuzzy convex. It is clear that $A \in \mathcal{K}(\Lambda)$ implies $\chi_A \in \mathcal{F}(\Lambda)$, while $A \in \mathcal{F}_c(\Lambda)$ implies $\chi_A \in \mathcal{F}_c(\Lambda)$, where χ_A is the indicator function of A.

A linear structure in $\mathcal{F}(\Lambda)$ is defined by the operation

$$(u+v)(x) = \sup_{y+z=x} min[u(y), v(z)]$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(x), \text{if } \lambda = 0. \end{cases}$$

where $u, v \in \mathcal{F}(\Lambda)$ and $\lambda \in R$.

A fuzzy random variable is defined as a function $X : \Omega \to \mathcal{F}(\Lambda)$ such that $L_\alpha X : \Omega \to \mathcal{K}(\mathcal{X}), L_\alpha X(\omega) = \{x \in \Lambda : X(\omega)(x) \geq \alpha\}$ is a random set for all $\alpha \in [0,1]$. A fuzzy random variable X is called integrably bounded if $L_\alpha X$ is integrably bounded for all $\alpha \in [0,1]$. Let $\Phi(\Omega, \Sigma, P, \Lambda) = \Phi$ be the set of all integrably bounded fuzzy random variables. With Φ_c we denote the set of all fuzzy random variables $X \in \Phi$ such that $L_\alpha X \in \mathcal{L}_c$ for all $\alpha \in (0,1]$.

4. Main Result

For $X, Y \in \Phi$, we define $D(X,Y) = \sup_{\alpha \geq 0} \Delta(L_\alpha X, L_\alpha Y)$. Two fuzzy random variables $X, Y \in \Phi$ are considered to be identical if $L_\alpha X = L_\alpha Y$ a.s. for all $\alpha \in [0,1]$. It is obvious that D is a metric in Φ and if

F,G are integrably bounded random set then $D(F,G) = \Delta(F,G)$.
To prove the main result, we need the following lemma.

Lemma 4.1. Let $\{F_\alpha : \alpha \in [0,1]\}$ be a family of random sets such that

(a) $F_0(\omega) = \Lambda$ for all $\omega \in \Omega$

(b) $\alpha \leq \beta$ implies $F_\beta \subseteq F_\alpha a.s.$

(c) $\alpha_1 \leq \alpha_2 \leq \ldots, \lim \alpha_n = \alpha$ implies $F_\alpha = \bigcap_{n=1}^{\infty} F_{\alpha_n}$ a.s.

Then the fuzzy random variable $X : \Omega \to \mathcal{F}(\mathcal{X})$ defined by $X(\omega)(x) = \sup\{\alpha \in [0,1] : x \in F_\alpha(\omega)\}$ has the property that $L_\alpha X = F_\alpha$ for every $\alpha \in [0,1]$.

Proof. It follows immediately from an application of lemma 1 [9].

Theorem 4.2. Φ is a complete metric space with repect to the metric D, and Φ_c is a closed subspace of Φ.

Proof. Let $\{X_n, n \geq 1\}$ be a Cachy sequence in Φ. Consider a fixed $\alpha > 0$. Then $\{L_\alpha(X_n), n \geq 1\}$ is a Cachy sequence in $\mathcal{L}$. Since $\mathcal{L}$ is complete with respect to Δ, it follows that

$$L_\alpha(X_n) \xrightarrow{\Delta} F_\alpha \in \mathcal{L}.$$

Actually, it is easy to see that $L_\alpha(X_n) \xrightarrow{\Delta} F_\alpha$ uniformly in $\alpha \in [0,1]$. Consider now the family $\{F_\alpha : \alpha \in [0,1]\}$, where $F_0(\omega) = \Lambda$ for all $\omega \in \Omega$.

Let $\varepsilon > 0$ and $\alpha \leq \beta$. Then

$$\delta(F_\beta, F_\alpha) \leq \delta(F_\beta, L_\beta(X_n) + \delta(L_\beta(X_n), L_\alpha(X_n)) + \delta(L_\alpha(X_n), F_\alpha)$$

Since $L_\beta(X_n) \subset L_\alpha(X_n)$, it follows that $\delta(L_\beta(X_n), L_\alpha(X_n)) = 0$.
Thus, $\delta(F_\beta, F_\alpha) \leq \delta(F_\beta, L_\beta(X_n)) + \delta(L_\alpha(X_n), F_\alpha) < \varepsilon$ if n is large enough. Hence $\delta(F_\beta, F_\alpha) = 0$ and since $F_\beta(\omega), F_\alpha(\omega)$ are closed, we have $F_\beta(\omega) \subseteq F_\alpha(\omega) a.s.$
Now take $\alpha > 0, \alpha_n \uparrow \alpha$. We have to show that

$$F_\alpha = \bigcap_{n=1}^{\infty} F_{\alpha_n} a.s.$$

It is clear that $F_\alpha \subseteq \bigcap_{n=1}^{\infty} F_{\alpha_n}$ a. s.
Using again the semimetric δ , we get for fixed j,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) + \delta(\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j), L_\alpha(X_j)) + \delta(L_\alpha(X_j), F_\alpha)$$

But $\delta(\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j), L_\alpha(X_j)) = 0$. Conseqently, for every $\varepsilon > 0$, there exits N_ε such that for $j \geq N_\varepsilon$

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq \varepsilon + \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j))$$

Now, for any $k \geq 1$,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) \leq \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_{\alpha_k}) + \delta(F_{\alpha_k}, L_{\alpha_k}(X_j)) + \delta(L_{\alpha_k}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)).$$

Since $\bigcap_{n=1}^{\infty} F_{\alpha_n} \subseteq F_{\alpha_k}$, we obtain

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) \leq \delta(M_{\alpha_k}, L_{\alpha_k}(X_j)) + \delta(L_{\alpha_k}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j))$$

Now $\delta(F_{\alpha_k}, L_{\alpha_k}(X_j)) < \varepsilon$ for $j \geq N_0$. Note that N_0 does not depend on k since the convergence $L_\alpha(X_j) \to F_\alpha$ is uniform. On the other hand, since $\{L_{\alpha_n}(X_j)\}$ decrease to $\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)$, it follows that $\delta(L_{\alpha_m}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) < \varepsilon$ for some m (depending on j). Thus, if j is large,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) < 2\varepsilon$$

Finally by taking j large enough, we obtain

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq 3\varepsilon$$

i.e.,

$$\bigcap_{n=1}^{\infty} F_{\alpha_n} \subseteq F_\alpha a.s.$$

Hence we obtain $\bigcap_{n=1}^{\infty} F_{\alpha_n} = F_\alpha a.s.$ Thus lemma4.1 is applicable and there exists $X \in \Phi$ with $L_\alpha(X) = F_\alpha$ for every $\alpha \in [0,1]$. It remains to show that $X_n \to X$ with respect to D. Let $\varepsilon > 0$. Then since $\{X_n\}$ is Cauchy, there exists N_ε such that $n, m > N_\varepsilon$ implies $D(X_n, X_m) < \varepsilon$. Let $n > N_\varepsilon$ be fixed. Then

$$\begin{aligned} D(L_\alpha(X_n), L_\alpha(X)) &= \lim_{m\to\infty} D(L_\alpha(X_n), L_\alpha(X_m)) \\ &\leq \overline{\lim_{m\to\infty}} \sup_{\alpha>0} D(L_\alpha(X_n), L_\alpha(X_m)) \\ &= \overline{\lim} D(X_n, X_m) < \varepsilon \end{aligned}$$

Thus,

$$D(X_n, X) = \sup_{\alpha>0} D(L_\alpha(X_n), L_\alpha(X)) \leq \varepsilon$$

for $n > N_\varepsilon$.

This completes the proof of the first statement and the second statement is trivial.

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