BOUNDARY OF MINKOWSKI ARC LENGTH IN MINKOWSKI PLANE

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1. Introduction

Chakerian, in [4], generalized Crofton's formula and Poincaré's formula in the Euclidean plane to them in Minkowski plane.

For a convex set $K$ in a Minkowski plane H.Flanders[5] proved the Bonnesen inequality in Minkowski plane:

$$\rho L - A - T \rho^2 \geq 0$$

for all $\rho$ in the interval $[r_{in}, r_{out}]$ where $L$ is Minkowski arc length, $A$ is Euclidean area, $T$ is Euclidean area of isoperimetrix of the Minkowski plane and $r_{in}$ and $r_{out}$ are inradius and outradius respectively.

In this paper, We develop arc length formula and area formula for the parallel set in a Minkowski plane. As an application we obtain boundary of the ratio of Minkowski arc length and Euclidean arc length.

2. Preliminaries

For a centrally symmetric closed convex curve $U$ enclosing area $\pi$ and with center at the origin $O$ of the Euclidean plane $R^2$ we shall assume throughout that $U$ is smooth and has positive finite curvature everywhere.

A usual norm $\| \cdot \|$ on $R^2$ defines a Minkowski metric, $m$, using the formula

$$m(x,y) = \frac{\|x - y\|}{r},$$

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where \( \|x - y\| \) is the Euclidean distance from \( x \) to \( y \), and \( r \) is the radius of \( U \) in the direction of vector \( x - y \). The set of points of \( \mathbb{R}^2 \), together with metric \( m \) is the Minkowskian plane, \( M^2 \). Certainly \( U \) is the unit ball in \( M^2 \) and it will be referred to as the indicatrix. Given a norm \( \ell(\cdot) \) on \( \mathbb{R}^2 \), one can define a Minkowski metric \( m \) using the formula

\[ m(x, y) = \ell(x - y) \]

so that unit ball is a convex set symmetric with respect to the origin.

To describe the Minkowski geometry associated with \( U \) and its relation to the Euclidean geometry of \( \mathbb{R}^2 \) we begin with two vectors \( e_1 = (\cos \theta, \sin \theta) \) and \( e_2 = (-\sin \theta, \cos \theta) \) which are orthonormal with respect to the Euclidean metric. Now let the boundary of \( U \) be described in polar coordinates by a function \( r(\theta) \). In searching for a substitute for the Frenet frame used in Euclidean geometry we set

\[ t(\theta) = r(\theta)e_1(\theta), \quad n(\theta) = \frac{1}{r(\theta)}e_2(\theta) - \left( \frac{1}{r(\theta)} \right)'e_1(\theta). \]

Then we have

\[ \frac{dt}{d\theta} = (r(\theta))^2n(\theta), \quad \frac{dn}{d\theta} = -h(\theta) + \frac{d^2h}{d\theta^2}t(\theta) \]

where \( h(\theta) = \frac{1}{r(\theta)} \).

The trace of \( n(\theta) \), \( 0 \leq \theta \leq 2\pi \), is a convex set \( I \), which is the so-called isoperimetrix, because it has the minimum boundary length (using the Minkowski definition of length) among all convex sets with a given area. (see [2] and [3].) It is easy to verify that \( I \) is polar reciprocal of \( U \), with respect to the Euclidean unit circle, rotated through deg 90. We shall always denote by \( T \) the area enclosed by \( I \). In terms of radial function \( r \) the function \( h = \frac{1}{r} \) is the support function for the isoperimetrix \( I \). Also \( I \) is up to homothety the unique convex shape which minimizes the Minkowski arc length of the boundary for a given enclosed area.

If \( X : [0, 1] \to \mathbb{R}^2 \) describes a differentiable curve, then
is the Minkowski length of the curve. The Minkowski element of arc length at any point is related to the Euclidean arc length by \( ds = r^{-1}da \).

3. Parallel Set and Geometric Inequalities in \( M^2 \)

**DEFINITION 1.** Given two bodies \( K \) and \( \tilde{K} \) the homothetic transformation of \( \tilde{K} \) and the Minkowski sum of \( K \) and \( \tilde{K} \) are the sets

\[
e\tilde{K} = \{ey|y \in K\} \quad \text{and} \quad K + \tilde{K} = \{x + y|x \in K, y \in \tilde{K}\} \]

respectively.

The set of convex bodies forms the positive cone of a vector space under these two operations. The "thickening" of \( K \) with respect to \( \tilde{K} \) is given by \( K + e\tilde{K} \) with epsilon positive. When \( \tilde{K} \) is the standard unit ball, this latter set is the set of all points in the plane whose distance from \( K \) is less than or equal to \( e \). The support function of the Minkowski sum satisfies

\[
h_{K + e\tilde{K}} = h_K + eh_{\tilde{K}}.
\]

While \( \tilde{K} \) remains fixed and centered at the origin, we shall frequently wish to translate the set \( K \). Translating \( K \) with respect to the origin corresponds to replacing \( h \) by \( h + acos\theta + bsin\theta \) for some \( a \) and \( b \). ([6]).

**DEFINITION 2.** Let \( K \) be a convex set of area \( A \) and Minkowskian perimeter \( L \) in a Minkowski plane with isoperimetric \( I \) containing area \( T \). Then \( \epsilon \)-parallel set is the set

\[
K_{\epsilon} = K + \epsilon I,
\]

Let \( K \) be an analytic closed convex curve which contains the origin in its interior. If \( h(\theta) \) is a support function of \( K \), then the radius of curvature of \( K \) at \( q \) is \( h(\theta) + h''(\theta) \) so that the euclidean line element
of \( K \) at \( q \) equals to \((h(\theta) + h''(\theta))d\theta\). Therefore the Minkowski length \( L(K) \) of \( K \) is

\[
L(K) = \int_0^{2\pi} (h(\theta) + h''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta
\]

where \( r(\theta) \) is the radial function for the indicatrix \( U \) of the Minkowski plane if the orientation of \( K \) is positive.

In the following theorem, we calculate Minkowskian perimeter and area of parallel set of convex set.

**Theorem 1.** Let \( K_t \) be a \( t \)-parallel set of a convex set \( K \). Then

\[
L(K_t) = L(K) + 2Tt, \quad A(K_t) = A(K) + L(K)t + Tt^2.
\]

where \( L \) denotes Minkowskian perimeter and \( A \) denotes Euclidean area.

**Proof.** The proof is a straightforward calculation. Let \( h(\theta) \) and \( p(\theta) \) be the support functions of \( K \) and \( I \) respectively. Then the support function of \( K_t \) is \( h_t(\theta) = h(\theta) + tp(\theta) \). So we have

\[
L(K_t) = \frac{1}{2} \int_0^{2\pi} (h_t(\theta) + h_t''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} (h(\theta) + tp(\theta) + h''(\theta) + tp''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} (h(\theta) + h''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta
\]

\[
+ \frac{t}{2} \int_0^{2\pi} (p(\theta) + p''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta
\]

\[
= L(K) + 2Tt
\]
and

\[ A(K_t) = \frac{1}{2} \int_0^{2\pi} \left( h_t^2(\theta) - (h_t'(\theta))^2 \right) d\theta \]

\[ = \frac{1}{2} \int_0^{2\pi} \left( h^2(\theta) - (h' (\theta))^2 \right) d\theta \]

\[ + t \int_0^{2\pi} (h(\theta)p(\theta) - h'(\theta)p'(\theta))d\theta \]

\[ + t^2 \frac{1}{2} \int_0^{2\pi} \left( p^2(\theta) - (p'(\theta))^2 \right) d\theta \]

\[ = A(K) + L(K)t + Tt^2. \]

**Theorem 2.** Let \( K \) be a convex set of perimeter \( L \) in a Minkowski plane \( M^2 \) with isoperimetrix \( I \). If we denote \( r_i \) and \( r_o \) by inradius and outradius of \( I \) respectively, then

\[ L_e r_i \leq L \leq L_e r_o \]

where \( L_e \) is Euclidean perimeter of \( K \) and \( T \) is area of isoperimetric.

**Proof.** Let \( D^i \) and \( D^o \) denote the Euclidean disks of radius \( r_i \) and \( r_o \) respectively. Then we have

\[ K + tD^i \subseteq K + tI \subseteq K + tD^o. \]

So we have

\[ A(K + tD^i) \leq A(K + tI) \leq A(K + tD^o). \]

So from (7) and (12) we have

\[ L_e r_i + \pi t r_i^2 \leq L + Tt \leq L_e r_o + \pi t r_o^2. \]

So if \( t \) tend to 0, then we have the desired inequality in (10).

From the Theorem 2 we have the following corollary.
COROLLARY 1. Let $K$ be a convex set with Minkowskian perimeter $L$ and Euclidean perimeter $L_e$ in a Minkowski plane $M^2$ with isoperimetrix $I$. If we denote $r_i$ and $r_o$ by inradius and outradius of isoperimetrix $I$ respectively, then $r_i \leq \frac{L}{L_e} \leq r_o$ and $\frac{L}{L_e} = 1$ if and only if $M^2$ is the Euclidean plane.

An easy corollary of the Crofton formula (Chakerian[4]) is that a convex hull of a closed simple curve has a boundary whose Minkowskian length is less than the Minkowskian length of the curve itself.

So we have the following corollary

COROLLARY 2. Let $C$ be an arbitrary closed curve in $M^2$, and $r_i$ and $r_o$ inradius and outradius of isoperimetrix $I$ respectively. If we denote the Minkowskian perimeter and Euclidean perimeter of convex hull of $C$ by $\bar{L}$ and $\bar{L}_e$ respectively, then

$$L \leq r_o^2 L_e, r_i^2 \bar{L}_e \leq L.$$

References


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