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BOUNDARY OF MINKOWSKI ARC LENGTH IN MINKOWSKI PLANE

Y. D. CHAI* AND YOUNG SOO LEE

1. INTRODUCTION

Chakerian, in [4], generalized Crofton's formula and Poincaré's formula in the Euclidean plane to them in Minkowski plane.

For a convex set K in a Minkowski plane H.Flanders[5] proved the Bonnesen inequality in Minkowski plane:
$\rho L - A - T\rho^2 \geq 0$ for all ρ in the interval $[r_{in}, r_{out}]$ where L is Minkowski arc length, A is Euclidean area, T is Euclidean area of isoperimetrix of the Minkowski plane and r_{in} and r_{out} are inradius and outradius respectively.

In this paper, We develop arc length formula and area formula for the parallel set in a Minkowski plane. As an application we obtain boundary of the ratio of Minkowski arc length and Euclidean arc length.

2. PRELIMINARIES

For a centrally symmetric closed convex curve U enclosing area π and with center at the origin O of the Euclidean plane R^2 we shall assume throughout that U is smooth and has positive finite curvature everywhere.
A usual norm $\|\cdot\|$ on R^2 defines a Minkowski metric, m, using the formula

$$m(x,y) = \frac{\|x-y\|}{r}, \tag{1}$$

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where $\|x-y\|$ is the Euclidean distance from x to y,and r is the radius of U in the direction of vector $x-y$. The set of points of R^2, together with metric m is the *Minkowskian plane*, M^2. Certainly U is the unit ball in M^2 and it will be referred to as the *indicatrix*. Given a norm $\ell(\cdot)$ on R^2, one can define a Minkowski metric m using the formula $m(x,y) = \ell(x-y)$ so that unit ball is a convex set symmetric with respect to the origin.

To describe the Minkowski geometry associated with U and its relation to the Euclidean geometry of R^2 we begin with two vectors $e_1 = (cos\theta, sin\theta)$ and $e_2 = (-sin\theta, cos\theta)$ which are orthonormal with respect to the Euclidean metric. Now let the boundary of U be described in polar coordinates by a function $r(\theta)$. In searching for a substitute for the Frenet frame used in Euclidean geometry we set

$$(2) \qquad t(\theta) = r(\theta)e_1(\theta), n(\theta) = \frac{1}{r(\theta)}e_2(\theta) - (\frac{1}{r(\theta)})'e_1(\theta).$$

Then we have

$$(3) \qquad \frac{dt}{d\theta} = (r(\theta))^2 n(\theta), \frac{dn}{d\theta} = -h(\theta)(h(\theta) + \frac{d^2h}{d\theta^2})t(\theta)$$

where $h(\theta) = \frac{1}{r(\theta)}$.

The trace of $n(\theta), 0 \le \theta \le 2\pi$, is a convex set I, which is the so-called isoperimetrix, because it has the minimum boundary length (using the Minkowski definition of length) among all convex sets with a given area.(see [2] and [3].) It is easy to verify that I is polar reciprocal of U, with respect to the Euclidean unit circle,rotated through deg 90. We shall always denote by T the area enclosed by I. In terms of radial function r the function $h = \frac{1}{r}$ is the support function for the isoperimetrix I. Also I is up to homothety the unique convex shape which minimizes the Minkowski arc length of the boundary for a given enclosed area.

If $X : [0,1] \to R^2$ describes a differentiable curve,then

$$L(X) = \int_0^1 \ell(X'(u))du = \int d\sigma \tag{4}$$

is the Minkowski length of the curve. The Minkowski element of arc length at any point is related to the Euclidean arc length by $d\sigma = r^{-1}ds$.

3. Parallel set and Geometric Inequalities in M^2

Definition 1. *Given two bodies K and $\tilde{K}$ the homothetic ,transformation of $\tilde{K}$ and the Minkowski sum of K and $\tilde{K}$ are the sets $\epsilon\tilde{K} = \{\epsilon y | y \in K\}$ and $K + \tilde{K} = \{x + y | x \in K, y \in \tilde{K}\}$ respectively.*

The set of convex bodies forms the positive cone of a vector space under these two operations. The "thickening" of K with respect to $\tilde{K}$ is given by $K + \epsilon\tilde{K}$ with epsilon positive. When $\tilde{K}$ is the standard unit ball, this latter set is the set of all points in the plane whose distance from K is less than or equal to ϵ. The support function of the Minkowski sum satisfies $h_{K+\epsilon\tilde{K}} = h_K + \epsilon h_{\tilde{K}}$. While $\tilde{K}$ remains fixed and centered at the origin,we shall frequently wish to translate the set K. Translating K with respect to the origin corresponds to replacing h by $h + acos\theta + bsin\theta$ for some a and b.([6]).

Definition 2. *Let K be a convex set of area A and Minkowskian perimeter L in a Minkowski plane with isoperimetrix I containing area T. Then ϵ-parallel set is the set*

$$K_\epsilon = K + \epsilon I. \tag{5}$$

Let K be an analytic closed convex curve which contains the origin in its interior. If $h(\theta)$ is a support function of K,then the radius of curvature of K at q is $h(\theta) + h''(\theta)$ so that the euclidean line element of K at q equals to $(h(\theta) + h''(\theta))d\theta$. Therefore the Minkowski length $L(K)$ of K is

$$L(K) = \int_0^{2\pi} (h(\theta) + h''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta \tag{6}$$

where $r(\theta)$ is the radial function for the indicatrix U of the Minkowski plane if the orientation of K is positive.

In the following theorem, we calculate Minkowskian perimeter and area of parallel set of convex set.

THEOREM 1. *Let K_t be a t-parallel set of a convex set K. Then*

$$L(K_t) = L(K) + 2Tt, A(K_t) = A(K) + L(K)t + Tt^2. \tag{7}$$

where L denotes Minkowskian perimeter and A denotes Euclidean area.

Proof. The proof is a straightforward calculation. Let $h(\theta)$ and $p(\theta)$ be the support functions of K and I respectively. Then the support function of K_t is $h_t(\theta) = h(\theta) + tp(\theta)$. So we have

$$\begin{aligned} L(K_t) &= \frac{1}{2}\int_0^{2\pi} (h_t(\theta) + h_t''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (h(\theta) + tp(\theta) + h''(\theta) + tp''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (h(\theta) + h''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta \\ &\quad + \frac{t}{2}\int_0^{2\pi} (p(\theta) + p''(\theta)) \frac{1}{r(\theta + \frac{\pi}{2})} d\theta \\ &= L(K) + 2Tt \end{aligned} \tag{8}$$

and

$$A(K_t) = \frac{1}{2}\int_0^{2\pi} (h_t^2(\theta) - (h_t'(\theta))^2)d\theta \tag{9}$$
$$= \frac{1}{2}\int_0^{2\pi} (h^2(\theta) - (h'(\theta))^2)d\theta$$
$$+ t\int_0^{2\pi} (h(\theta)p(\theta) - h'(\theta)p'(\theta))d\theta$$
$$+ t^2\frac{1}{2}\int_0^{2\pi} (p^2(\theta) - (p'(\theta))^2)d\theta$$
$$= A(K) + L(K)t + Tt^2.$$

THEOREM 2. *Let K be a convex set of perimeter L in a Minkowski plane M^2 with isoperimetrix I. If we denote r_i and r_o by inradius and outradius of I respectively, then*

$$L_e r_i \leq L \leq L_e r_o \tag{10}$$

where L_e is Euclidean perimeter of K and T is area of isoperimetrix.

Proof. Let D^i and D^o denote the Euclidean disks of radius r_i and r_o respectively. Then we have

$$K + tD^i \subseteq K + tI \subseteq K + tD^o. \tag{11}$$

So we have

$$A(K + tD^i) \leq A(K + tI) \leq A(K + tD^o). \tag{12}$$

So from (7) and (12) we have

$$L_e r_i + \pi t {r_i}^2 \leq L + Tt \leq L_e r_o + \pi t {r_o}^2. \tag{13}$$

So if t tend to 0, then we have the desired inequality in (10).

From the Theorem 2 we have the following corollary.

COROLLARY 1. *Let K be a convex set with Minkowskian perimeter Land Euclidean perimeter L_e in a Minkowski plane M^2 with isoperimetrix I.If we denote r_i and r_o by inradius and outradius of isoperimetrix I respectively,then $r_i \leq \frac{L}{L_e} \leq r_o$ and $\frac{L}{L_e} = 1$ if and only if M^2 is the Euclidean plane.*

An easy corollary of the Crofton formula (Chakerian[4]) is *that a convex hull of a closed simple curve has a boundary whose Minkowskian length is less than the Minkowskian length of the curve itself.*

So we have the following corollary

COROLLARY 2. *Let C be an arbitrary closed curve in M^2, and r_i and r_o inradius and outradius of isoperimetrix I respectively. If we denote the Minkowskian perimeter and Euclidean perimeter of convex hull of C by $\tilde{L}$ and $\tilde{L}_e$ respectively, then*

$$\tilde{L} \leq r_o^2 L_e, r_i^2 \tilde{L}_e \leq L. \tag{14}$$

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Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746
South Korea