Pusan Kyŏngnam Math J 10(1994), No 2, pp 261-263

DIRECT PROJECTIVE MODULES WITH THE SUMMAND INTERSECTION PROPERTY

HAN, CHANG WOO, LEE, BU YOUNG AND CHOI, SU JEONG

1. Introduction

Throughout this paper, R is a ring with identity and all modules are unitary R-modules. We denote the endomorphism ring of M by End(M). The module M is said to be quasi-projective if, given an R-homomorphism $g : M \longrightarrow L$, for each epimorphism $\alpha : M \longrightarrow L$, there exists an endomorphism h of M such that $\alpha \circ h = g$. The module M is said to be direct projective if, given any direct summand A of M and $\pi : M \longrightarrow A$ a projection map, for each epimorphism $\alpha : M \longrightarrow A$, there exists an endomorphism ψ of M such that $\alpha \circ \psi = \pi$. The concept of direct projectivity is a generalization of quasi-projectivity. The module M has the summand intersection property if the intersection of two direct summands is again a direct summand. Kaplansky observed that if F is a free module over a principal ideal domain, then the intersection of any two direct summands of F is again a direct summand.

In this paper, we consider direct projective modules with the summand intersection property and obtain several conditions so that a direct projective module has the summand intersection property. As a result, we have some properties of a direct projective module.

THEOREM 1.1 [1]. *The following properties of the module M are equivalent.*

(i) M is direct projective.

(ii) Every exact sequence $N \longrightarrow A \longrightarrow O$ with N an epimorphic image of M and A a direct summand of M splits.

Received September 5,1994.

This paper was supported by research fund of Dong-A University, 1994 .

THEOREM 1.2 [2]. *the module M has the summand intersection property if and only if, for every decomposition $M = A \oplus B$ and every $\varepsilon : A \longrightarrow B$, the kernel of ε is a direct summand of A.*

2. Results

THEOREM 2.1. *Let M be a direct projective module. If for every decomposition $M = A \oplus B$ and every $\varepsilon : A \longrightarrow B$, Im ε is a direct summand of M, then M has the summand intersection property.*

Proof. For every decomposition $M = A \oplus B$ and every $\varepsilon : A \longrightarrow B$, assume that Im ε is a direct summand of M. It is sufficient to show that Ker ε is a direct summand of A. A is an epimorphic image of M. Since M is direct projective, by applying Theorem 1.1, an exact sequence $O \longrightarrow \text{Ker } \epsilon \longrightarrow A \longrightarrow \text{Im } \varepsilon \longrightarrow O$ splits. This implies Ker ε is a direct summand of A. Hence M has the summand intersection property.

THEOREM 2.2. *If $M \oplus L$ has the summand intersection property for all the module L, then the module M is quasi-projective.*

Proof. Assume that $M \oplus L$ has the summand intersection property for all the module L. Then by Theorem 1.2, every exact sequence $M \xrightarrow{f} L \longrightarrow O$ splits. Therefore there exists an R-homomorphism $f' : L \longrightarrow M$ such that $f \circ f' = I_L$. For given $g : M \longrightarrow L$, let $h = f' \circ g$. Then $f \circ h = g$. hence M is quasi-projective.

THEOREM 2.3. *If every submodule of a direct projective module M is direct projective, then M has the summand intersection property.*

Proof. For every decomposition $M = A \oplus B$ and every $\varepsilon : A \longrightarrow B$, $A \oplus \text{Im } \varepsilon$ is a submodule of M, and $A \oplus \text{Im } \varepsilon$ is direct projective. Clearly A is an epimorphic image of M. According to Theorem 1.1, an exact sequence $O \longrightarrow \text{Ker } \varepsilon \longrightarrow A \longrightarrow \text{Im } \varepsilon \longrightarrow O$ splits. Hence by Theorem 1.2, M has the summand intersection property.

THEOREM 2.4. *Let M be direct projective. If End(M) is a regular ring, then M has the summand intersection property.*

Proof. Let End(M) be a regular ring and consider every $f : A \oplus B \longrightarrow B \oplus A$ by setting $f = (f_1, f_2)$, where $f_1 : A \longrightarrow B,\ f_2 : B \longrightarrow A$ are R-homomorphisms. Then Im f and Ker f are direct summands of M.[4, Lemma 3.1] It follows that Ker f_1 is a direct summand of A. Hence by Theorem 1.2, M has the summand intersection property.

THEOREM 2.5. *If every finitely generated direct projective module has the summand intersection property, then R is a semihereditary ring.*

Proof. Suppose that all finitely generated direct projective modules have the summand intersection property. Let A be a finitely generated ideal of R, $p : R^n \longrightarrow A$ an epimorphism and $i : A \longrightarrow R$ a canonical inclusion map. Since R^{n+1} has the summand intersection property, we see from Theorem 1.2 that ker $(i \circ p)$ is a direct summand of R^n. Hence A is projective module. This means that R is a semihereditary ring.

References

1 J Hausen, *Direct projective modules*, Bull Inst Math Acad Sinica **9** (1981), 447-451.

2. J Hausen, *Modules with the summand intersection property*, Comm. Algebra **17** (1989), 135-148.

3 J.J. Rotman, *An introduction to homological algebra*, Academic press New York (1979).

4. R Ware, *Endomorphism rings of projective modules*, Trans Amer. Math. Soc **155** (1971), 233-256.

5. G.V Wilson, *Modules with the summand intersection property*, Comm. Algebra **14** (1986), 21-38.

Department of Mathematics
Dong-A University
Busan 604-714, Korea