DIRECT PROJECTIVE MODULES WITH THE SUMMAND INTERSECTION PROPERTY

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1. Introduction

Throughout this paper, \( R \) is a ring with identity and all modules are unitary \( R \)-modules. We denote the endomorphism ring of \( M \) by \( \text{End}(M) \). The module \( M \) is said to be quasi-projective if, given an \( R \)-homomorphism \( g : M \to L \), for each epimorphism \( \alpha : M \to L \), there exists an endomorphism \( h \) of \( M \) such that \( \alpha \circ h = g \). The module \( M \) is said to be direct projective if, given any direct summand \( A \) of \( M \) and \( \pi : M \to A \) a projection map, for each epimorphism \( \alpha : M \to A \), there exists an endomorphism \( \psi \) of \( M \) such that \( \alpha \circ \psi = \pi \). The concept of direct projectivity is a generalization of quasi-projectivity. The module \( M \) has the summand intersection property if the intersection of two direct summands is again a direct summand. Kaplansky observed that if \( F \) is a free module over a principal ideal domain, then the intersection of any two direct summands of \( F \) is again a direct summand.

In this paper, we consider direct projective modules with the summand intersection property and obtain several conditions so that a direct projective module has the summand intersection property. As a result, we have some properties of a direct projective module.

**Theorem 1.1** [1]. The following properties of the module \( M \) are equivalent.

(i) \( M \) is direct projective.

(ii) Every exact sequence \( N \to A \to O \) with \( N \) an epimorphic image of \( M \) and \( A \) a direct summand of \( M \) splits.
**Theorem 1.2** [2]. The module $M$ has the summand intersection property if and only if, for every decomposition $M = A \oplus B$ and every $\varepsilon : A \rightarrow B$, the kernel of $\varepsilon$ is a direct summand of $A$.

2. Results

**Theorem 2.1.** Let $M$ be a direct projective module. If for every decomposition $M = A \oplus B$ and every $\varepsilon : A \rightarrow B$, $\text{Im} \, \varepsilon$ is a direct summand of $M$, then $M$ has the summand intersection property.

**Proof.** For every decomposition $M = A \oplus B$ and every $\varepsilon : A \rightarrow B$, assume that $\text{Im} \, \varepsilon$ is a direct summand of $M$. It is sufficient to show that $\text{Ker} \, \varepsilon$ is a direct summand of $A$. $A$ is an epimorphic image of $M$. Since $M$ is direct projective, by applying Theorem 1.1, an exact sequence $0 \rightarrow \text{Ker} \, \varepsilon \rightarrow A \rightarrow \text{Im} \, \varepsilon \rightarrow O$ splits. This implies $\text{Ker} \, \varepsilon$ is a direct summand of $A$. Hence $M$ has the summand intersection property.

**Theorem 2.2.** If $M \oplus L$ has the summand intersection property for all the module $L$, then the module $M$ is quasi-projective.

**Proof.** Assume that $M \oplus L$ has the summand intersection property for all the module $L$. Then by Theorem 1.2, every exact sequence $M \rightarrow L \rightarrow O$ splits. Therefore there exists an $R$-homomorphism $f' : L \rightarrow M$ such that $f \circ f' = I_L$. For given $g : M \rightarrow L$, let $h = f' \circ g$. Then $f \circ h = g$. Hence $M$ is quasi-projective.

**Theorem 2.3.** If every submodule of a direct projective module $M$ is direct projective, then $M$ has the summand intersection property.

**Proof.** For every decomposition $M = A \oplus B$ and every $\varepsilon : A \rightarrow B$, $A \oplus \text{Im} \, \varepsilon$ is a submodule of $M$, and $A \oplus \text{Im} \, \varepsilon$ is direct projective. Clearly $A$ is an epimorphic image of $M$. According to Theorem 1.1, an exact sequence $O \rightarrow \text{Ker} \, \varepsilon \rightarrow A \rightarrow \text{Im} \, \varepsilon \rightarrow O$ splits. Hence by Theorem 1.2, $M$ has the summand intersection property.
THEOREM 2.4. Let $M$ be direct projective. If $\text{End}(M)$ is a regular ring, then $M$ has the summand intersection property.

Proof. Let $\text{End}(M)$ be a regular ring and consider every $f : A \oplus B \to B \oplus A$ by setting $f = (f_1, f_2)$, where $f_1 : A \to B$, $f_2 : B \to A$ are $R$-homomorphisms. Then $\text{Im} f$ and $\text{Ker} f$ are direct summands of $M$. [4, Lemma 3.1] It follows that $\text{Ker} f_1$ is a direct summand of $A$. Hence by Theorem 1.2, $M$ has the summand intersection property.

THEOREM 2.5. If every finitely generated direct projective module has the summand intersection property, then $R$ is a semihereditary ring.

Proof. Suppose that all finitely generated direct projective modules have the summand intersection property. Let $A$ be a finitely generated ideal of $R$, $p : R^n \to A$ an epimorphism and $i : A \to R$ a canonical inclusion map. Since $R^{n+1}$ has the summand intersection property, we see from Theorem 1.2 that $\ker (i \circ p)$ is a direct summand of $R^n$. Hence $A$ is projective module. This means that $R$ is a semihereditary ring.

References


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