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A NOTE ON JOINTLY CENTRALOID OPERATORS

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1.Introduction

Let $B(H)$ be the algebra of bounded operators on a complex Hilbert space H and $A = (A_1, A_2, \ldots, A_n)$ be an n-tuple of operators on H. By an operator-family we shall mean a commuting n-tuple of operators and denote the set of all operator-familes by $B^n(H)$. We shall say that a point $z = (z_1, \cdots, z_n)$ of $\mathbb{C}^n$ is in Taylor's joint spectrum $S_p(A)$ of A if $z - A = (z_1 - A_1, \ldots, z_n - A_n)$ is singular. The joint numerical range $W(A)$ of A is the subset of $\mathbb{C}^n$ such that $W(A) = \{((A_1x, x), \cdots, (A_nx, x)) : x \in H, \|x\| = 1\}$. The joint norm,joint spectral radius and joint numerical radius of A , denote by $\|A\|$, $r(A)$ and $w(A)$ respectively,are defined by

$$\|A\| = \sup\{(\sum_{k=1}^{n} \|A_kx\|^2)^{\frac{1}{2}} : \|x\| = 1\},$$

$$r(A) = \sup\{(\sum_{k=1}^{n} |z_k|^2)^{\frac{1}{2}} : (z_1, z_2, \cdots, z_n) \in S_p(A)\} \quad \text{and}$$

$$w(A) = \sup\{(\sum_{k=1}^{n} |(A_kx, x)|^2)^{\frac{1}{2}} : \|x\| = 1\} \quad \text{respectively.}$$

We shall abbreviate $((A_1x, x), \cdots, (A_nx, x))$ and $(\sum_{k=1}^{n} \|A_kx\|^2)^{\frac{1}{2}}$ to (Ax, x) and $\|Ax\|$, respectively. It is wll known $\|A\| \geq w(A) \geq r(A)$.

We shall introduce some classes of operator-familes. An opertator-family A is called jointly noramloid, jointly transloid, jointly spectraloid and jointly convexoid,respectively if $\|A\| = w(A)$,$A - z$ is jointly normaloid for any point z, $w(A) = r(A)$ and $\text{Co}S_p(A) = \text{Co}\overline{W(A)}$,respectively.

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Let R_A and W_A(resp. z_A and w_A) be the jointly radius (resp.center) of the smallest disc containing $S_p(A)$ and $W(A)$ of A. Then these radius are translatable in the sense that $R_{A-z} = R_A$ and $W_{A-z} = W_A$,respectively for every complex number z. Obviously,R_A and W_A corresponds to $R(A)$ and $w(A)$ respectively. In [4],we introduced the jointly centroid operator as follows :

An opertor family $A(\in B^n(H))$ is called jointly centroid if $A - z_A$ is jointly normaloid. In [2],Fan Ming showed that $A(\in B^n(H))$ satisfies the inequality $\sup\{\|Ax\|^2 - |(Ax,x)|^2\} \geq R_A^2$. Moreover, if A is jointly transloid, then the above equality holds. In [4], we introduced the jointly transcendentral radius M_A as $M_A = \sqrt{B_A}$ where

$B_A = \sup_{\|x\|=1}\{\|Ax\|^2 - |(Ax,x)|^2\}$which was due to Garske [3].

In Takaguchi [5], the center m_A of mass for an n-tuple of operators A has been defined and stated that the center m_A of A is coincident with the center of the smallest sphere containg of $S_p(A)$ in the case of A being jointly transloid. In this note, We shall modify that the Fan Ming's theorem [2] and introduce a new class of operator-families called jointly centraloid, which includes both a classes of jointly-convexoid operayor-families and a class of jointly centroid operator-families. Moreover, we shall observe that the class of jointly centraloid operator-familes includes some kinds of classes of operators-familes.

2. Jointly Transcendental radius and jointly centraloid operators

At first, we shall show that M_A is translatable. Since

$$\sum_{k=1}^{n}(\|A_kx\|^2 - |(A_kx,x)|^2) = \sum_{k=1}^{n}\{\|(A_k - z_k)x\|^2 - |((A_k - z_k)x,x)|^2\}$$

for all $z \in \mathbb{C}^n$, we have $M_{A-z} = M_A$. In [4],$\|A - m_A\| = M_A$ and if A is an n-tuple of operators, then there exists a unique $m_A \in \mathbb{C}^n$ such that $\|A - m_A\|^2 + |z|^2 \leq \|(A - m_A) + z\|^2$ for all $z \in \mathbb{C}^n$.

Hence the jointly transcendental radius is nothing but the distance between A and scalars. We shall prove the following modification of the Fan Ming's theorem [2]:

THEOREM 1. *The jointly transcendental closed ball of A contains* $W(A)$ *and* $S_p(A)$, *so* $R_A \leq W_A \leq M_A$.

Proof. Since M_A and W_A are translatable, we may assume that $m_A = 0$, that is $M_A = \|A\|$. Hence the jointly transcendental closed ball centered the orgin with radius $\|A\|$ clearly contains $W(A)$. Since R_A and W_A correspond to $r(A)$ and $w(A)$ respectively, $R_A \leq W_A$, and so $R_A \leq W_A \leq M_A$.

REMARK.

(1) If $R_A = M_A$, then A is a jointly centroid operator-family.

(2) By the uniqueness of the smallest disc containing $W(A)$, we can deduce that $W_A < M_A$ unless $w_A = m_A$.

(3) If $M_A = W_A$, then $m_A = w_A$.

(4) A jointly translod operator-family is jointly centroid.

We shall introduce a jointly centraloid operato-family as follows:
$A(\in B^n(H))$ is jointly centraloid if $A - z_A$ is jointly spectraloid.

LEMMA 2 [2]. *An operator-family A is jointly convexoid if and only if* $A - z$ *is jointly spectraloid for all* $z = (z_1, \cdots, z_n)$.

From Lemma 2, every jointly convexoid operator-family is jointly centraloid.

LEMMA 3 [4]. *An operator-family A is jointly centroid if and only if* $B_A = R_A^2$.

The following characterization of centraloid operator-families is in parallel with one of jointly centroid operator-families given in [4]:

THEOREM 4. *An operator-family A is jointly centraloid if and only if* $W_A = R_A$.

Proof. Since $R_A = r(A - z_A)$, if A is jointly centraloid,then $w(A - z_A) = r(A - z_A) = R_A$. But we have $W_A < w(A - z)$ for $z \neq w_A$ and $W_A \leq w(A - z_A) = R_A$. Thus it follows from Theorem 1 that $W_A = R_A$. Conversely, if $W_A = R_A$,then $r(A - w_A) = w(A - w_A) = W_A = R_A \leq r(A - w_A)$, so $r(A - w_A) = R_A$.

COROLLARY 5. *Every jointly centroid operator-family is jointly centraloid.*

Proof. It is obviously by Lemma 3 and Theorem 4.

COROLLARY 6. *An operator-family A satisfies $M_A = W_A$ if and only if $A - w_A$ is jointly nomaloid.*

Proof. Since $W_A \leq w(A - m_A) \leq \|A - m_A\| = M_A$, if $M_A = W_A$, then $W_A = w(A - m_A)$ and $w(A - m_A) = \|A - m_A.\|$ Hence $A - m_A$ is jointly normaloid. Since $m_A = w_A$, $A - w_A$ is jointly normaloid. Conversely, if $A - w_A$ is jointly normaloid,then $M_A \leq \|A - w_A\| = w(A - w_A) = W_A \leq M_A$. Thus $W_A = M_A$.

From Lemma 2,3 and Theorem 4, we shall show that the following:

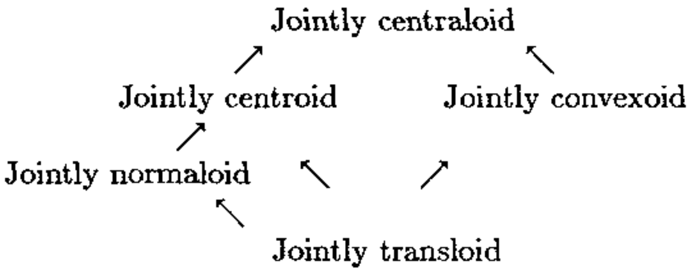
REMARK.

(1) There is a jointly centraloid operator-family which is not jointly centroid.

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Then $A = (A_1, A_2)$ is a commuting pair of operator A_1 and A_2, $S_p(A) = \{(1,0),(1,1)\}$, and $R_A = \frac{1}{2}$. Also we have $W_A = \frac{1}{2}$ for the unit vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 \geq x_2 \geq 0$, While $B_A = 1 > \frac{1}{4} = R_A^2$. Thus A is jointly centraloid but jointly centroid.

(2) There is a jointly centraloid operator-family which is not jointly convexoid. Let A_1 be the 4×4 identity matrix and $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A = (A_1, A_2)$ is a commuting pair of operators A_1 and A_2,$S_p(A) = \{(1,0),(1,1),(1,-1)\}$, $z_A = (1,0)$, and $R_A^2 = 1 = B_A$. Thus A is jointly centraloid but $A - z$ is not jointly spectraloid for any $z \in \mathbb{C}^2$, so that A is not jointly convexoid.

From Remark,Lemma 2, Corollary 5 and in [4], we shall show that the class of jointly centraloid operator-framilies includes the following kinds of classes of operator-families:



,where the symbol $\rightarrow$ indicate the inclusion relation.

References

1. M.Cho and M.Takaguchi, *Some classes of commmuting n-tuple of operators*, Studia Math. **802** (1984), 245–259.
2. M.Fan, *Garske's inequality for an n-tuple of operators*, Integral Equation and Operator Theory **14** (1991), 787–793.
3. G.Garske, *An equality cocerning the smallest disc that contains the spectrum of an operator*, Proc.A M.S. **78(4)** (1980), 529–532
4. H.G. Je,Y.S. Park and C.S. Ryoo, *On the properties of jointly operators*, Pusan Kyŏngnam Math.J. **9(2)** (1993), 251–260
5. M.Takaguchi, *Centers of mass for operator-familes*, Glasgow Math.J **34** (1992), 123–126.

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