ON A CONJECTURE OF GRAHAM

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Let $A$ be a finite sequence of $n$ positive integers $a_1 < a_2 < ... < a_n$. Graham [2] has conjectured that $\max\{a_i/(a_i, a_j)\} \geq n$. The conjecture has been verified in some special cases (see references).

In this paper we verify the conjecture in three cases.

A prime $p$ is called a prime factor of $A$ if $p$ is a prime factor of some $a_i$. Denote by $P(a_i)$ (resp. $P(A)$) the set of all prime factors of $a_i$ (resp. $A$).

**THEOREM 1.** If $P(a_k) = \{p_1, p_2, ..., p_s\}$ for some $a_k$, and $a_i \neq p_1^{i_1}p_2^{i_2}...p_s^{i_s}a_j$ for $i \neq j$, $d_1, d_2, ..., d_s \geq 0$, then $\max\{a_i/(a_i, a_j)\} \geq n$.

**Proof.** Let $a_i = p_1^{i_1}p_2^{i_2}...p_s^{i_s}b_i$ for all $a_i$ where $i_1, i_2, ..., i_s \geq 0$ and $p_1, p_2, ..., p_s \mid b_i$. Then $b_i \neq b_j$ for $i \neq j$ and $b_k = 1$. So $\max\{b_i/(b_i, b_j)\} \geq n$. Since

$$a_i/(a_i, a_j) = (p_1^{i_1}p_2^{i_2}...p_s^{i_s}b_i)/(p_1^{j_1}p_2^{j_2}...p_s^{j_s}b_1, p_1^{j_1}p_2^{j_2}...p_s^{j_s}b_j)$$

$$= (p_1^{i_1}p_2^{i_2}...p_s^{i_s}b_i)/((p_1^{i_1}p_2^{i_2}...p_s^{i_s}, p_1^{j_1}p_2^{j_2}...p_s^{j_s})(b_i, b_j))$$

$$\geq b_i/(b_i, b_j)$$

for all $i, j$, it follows that $\max\{a_i/(a_i, a_j)\} \geq \max\{b_i/(b_i, b_j)\} \geq n$.

**COROLLARY 2.** Suppose that $A$ contains a prime power $p^d$ with the property: $a_i \neq p^k a_j$ for $i \neq j$ and $k \geq 0$. Then $\max\{a_i/(a_i, a_j)\} \geq n$.

Following corollary 2, we have

**COROLLARY 3 ([5]).** Let $A$ be $p$-simple for a prime $p \neq 2$ and suppose that $A$ contains a prime power $a_k = p^d$ ($d \geq 0$). Then $\max\{a_i/(a_i, a_j)\} \geq n$.

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LEMMA 4 ([4]). If $F$ is a finite collection of sets then the number of distinct differences of members of $F$ is at least as large as the number of members of $F$.

THEOREM 5. Let $P(A) = \{p_1, p_2, \ldots, p_m\}$ and let $a_i = p_1^{i_1} p_2^{i_2} \cdots p_m^{i_m}$ for all $a_i$. If nonzero numbers of $\{1, 2, \ldots, n\}$ are equal for all $p_j$, then $\max_{i,j} \{a_i/(a_i, a_j)\} \geq n$.

Proof. Let $F_i = \{p \in P(A) : p | a_i\}, i = 1, 2, \ldots, n$. Then $a_i = a_j$ if and only if $F_i = F_j$ for all $i, j$. Clearly $F_1, F_2, \ldots, F_n$ are $n$ different sets. It follows from Lemma 4 that the number of different members of $\{F_i \setminus F_j : i, j = 1, 2, \ldots, n\}$ is at least as large as $n$. Since $F_i \setminus F_j = \{p \in P(A) : p | a_i/(a_i, a_j)\}$ and since $F_i \setminus F_j = F_h \setminus F_k$ if and only if $a_i/(a_i, a_j) = a_k/(a_k, a_k)$ for $1 \leq i, j, h, k \leq n$, it follows that $\{a_i/(a_i, a_j) : i, j = 1, 2, \ldots, n\}$ contains at least $n$ different numbers. Thus, $\max_{i,j} \{a_i/(a_i, a_j)\} \geq n$.

COROLLARY 6 ([4]). If the members of $A$ are squarefree integers then $\max_{i,j} \{a_i/(a_i, a_j)\} \geq n$.

THEOREM 7. If $[a_1, a_2, \ldots, a_n] \notin [1, 2, \ldots, n]$, then

$$\max_{i,j} \{a_i/(a_i, a_j)\} > n$$

where $[a_1, a_2, \ldots, a_n] = \text{lcm}\{a_1, a_2, \ldots, a_n\}$.

Proof. Suppose $[a_1, a_2, \ldots, a_n] \notin [1, 2, \ldots, n]$. Then there exists some $a_k$ such that $a_k \notin [1, 2, \ldots, n]$. Let $P(A) = \{p_1, p_2, \ldots, p_m\}$ and $a_k = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Then there exists $p_s^{k_s}$ such that $p_s^{k_s} \notin [1, 2, \ldots, n]$. Hence $p_s^{k_s} > n$. Since $(a_1, a_2, \ldots, a_n) = 1$, we have $(p_s, a_t) = 1$ for some $a_t$, and hence $(p_s^{k_s}, a_t) = 1$. Thus $a_k/(a_k, a_t) \geq p_s^{k_s} > n$, and therefore, $\max_{i,j} \{a_i/(a_i, a_j)\} > n$.

References
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