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SOBOLEV'S LEMMA AND THE SPACES $\mathcal{D}_p(\Omega)$.

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0. Introduction

It is well known that if $p(x) = \ln(1 + |x|)$, then the space $\mathcal{D}_p$ coincides with $\mathcal{D} = C_0^\infty(R^n)$. We can show the above result by Sobolev's Lemma. Also we have the same results for the Beurilng's generalized distributions.

1. Definitions and notations

The *normalized Lebesgue measure* on R^n is the measure m_n defined by $dm_n(x) = (2\pi)^{-n/2}dx$. The usual Lebesgue spaces L^p, or $L^p(R^n)$, will be normed by means of m_n:

$$||f||_{L^p} = \{\int_{R^n} |f|^p dm_n\}^{1/p} \qquad (1 \leq p < \infty).$$

For each $t \in R^n$, the *character* e_t is the function defined by

$$e_t(x) = e^{itx} = \exp\{i(t_1x_1 + \cdots + t_nx_n)\} \qquad (x \in R^n).$$

The *Fourier transform* of the function $f \in L^1(R^n)$ is the function $\hat{f}$ defined by

$$\hat{f}(t) = \int_{R^n} fe_{-t}dm_n \qquad (t \in R^n).$$

The relation $S_1 \Subset S_2$ shall mean that the closure of S_1 is compact and contained in the interior of S_2. If $\{S_j\}_{j=1}^\infty$ is a sequence of sets, the relation $S_j \nearrow\nearrow S$ shall mean that $S_j \Subset S_{j+1}(j = 1, 2, \cdots)$ and that $S = \cup S_j$. Let p be a real-valued function on R^n, continuous at the origin and having the property

$$(\alpha) \qquad 0 = p(0) = \lim_{x \to 0} p(x) \leq p(\xi + \eta) \leq p(\xi) + p(\eta) \quad (^\forall \xi, \eta \in R^n).$$

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DEFINITION 1.1. Let $\mathcal{M}_0 = \mathcal{M}_0(n)$ be the set of all continuous real-valued functions p on R^n satisfying the conditions (α) and

$$(\beta) \qquad J_n(p) = \int_{|\xi|\geq 1} \frac{p(\xi)}{|\xi|^{n+1}} d\xi < \infty.$$

DEFINITION 1.2. Let p satisfy (α). If $\phi \in L^1(R^n)$ and if λ is a real number, we write

$$\|\phi\|_\lambda = \|\phi\|_\lambda^{(p)} = \int |\hat{\phi}(\xi)| e^{\lambda p(\xi)} d\xi.$$

Let $\mathcal{D}_p$ be the set of all ϕ in $L^1(R^n)$ such that ϕ has compact support and $\|\phi\|_\lambda < \infty$ for all $\lambda > 0$. The elements of $\mathcal{D}_p$ will be called test functions.

DEFINITION 1.3. Let p_1 and p_2 be the elements in $\mathcal{M}_0(n)$. If for some real a and positive b we have $p_2(\xi) \leq a + bp_1(\xi) \quad (^\forall \xi \in R^n)$. Then p_2 is said to be *dominated* by p_1 with some constant translation. We denote this by $p_2 \prec p_1$.

DEFINITION 1.4. If K is a compact subset of R^n, $\mathcal{D}_p(K) = \{\phi \in \mathcal{D}_p; supp\phi \subset K\}$. Note that the space $\mathcal{D}_p(K)$ is a Fréchet space under the natural linear structure and the seminorms $\|\cdot\|_m \quad (m = 1, 2, \cdots)$.

DEFINITION 1.5. If Ω is an open subset of R^n and if $K_\nu \nearrow\!\!\nearrow \Omega$ we define $\mathcal{D}_p(\Omega)$ as the inductive limit of the Fréchet spaces $\mathcal{D}_p(K_\nu)$, i.e., $\mathcal{D}_p(\Omega) = \operatorname{ind\,lim}_{K_\nu \Subset \Omega} \mathcal{D}_p(K_\nu)$.

DEFINITION 1.6. Let $\mathcal{M} = \{p \in \mathcal{M}_0(n) : p \text{ satisfy condition } (\gamma)\}$:

$$(\gamma) \qquad p_0 \prec p, \quad \text{where} \quad p_0(x) = \ln(1 + |x|) \quad (x \in R^n).$$

2. Sobolev's Lemma

The elements of the dual space $\mathcal{D}'_p(\Omega)$ will be called Beurling's generalized distributions. Here we call them simply generalized distributions.

DEFINITION 2.1. A complex measurable function f, defined in an open set $\Omega \subset R^n$, is said to be ***locally*** L^2 ***in*** Ω if $\int_K |f|^2 dm_n < \infty$ for every compact $K \subset \Omega$.

DEFINITION 2.2. A distribution (resp. generalized distribution) $u \in \mathcal{D}'(\Omega)$ (resp $\mathcal{D}'_p(\Omega)$) is ***locally*** L^2 if there is a function g, locally L^2 in Ω, such that $u(\phi) = \int_\Omega g\phi dm_n$ for every $\phi \in \mathcal{D}(\Omega)$ (resp. $\mathcal{D}_p(\Omega)$). To say that a function f has a distribution (resp. generalized distribution) derivative $D^\alpha f$ which is locally L^2 refers to the ***distribution*** (resp. ***generalized distribution***) $D_\alpha f$ and means, explicitly, that there is a function g, locally L^2, such that

$$\int_\Omega g\phi dm_n = (-1)^{|\alpha|} \int_\Omega f D^\alpha \phi dm_n$$

for every $\phi \in \mathcal{D}(\Omega)$ (resp. $\mathcal{D}_p(\Omega)$).

We shall write D_i^k for the differential operator $(\partial/\partial x_i)^k$.

LEMMA 2.3. *Suppose* n, q, r *are integers, with* $n > 0, q \geq 0$, *and* $2r > 2q + n$. *Suppose* f *is a function in an open set* $\Omega \subset R^n$, *where distribution (resp. generalized distribution) derivatives* D_i^k *ars locally* L^2 *in* Ω, *for* $1 \leq i \leq n, \quad 0 \leq k \leq r$. *Then there exists a function* $F \in (L^1 \cap L^2)(R^n)$ *satisfying the following :*

(1) $f = F$ *in an open set* ω *such that* $\omega \Subset \Omega$.

(2) $\int_{R^n} (1 + |y|)^{2r} |\hat{F}(y)|^2 dm_n(y) < \infty$.

(3) $\int_{R^n} (1+|y|)^q |\hat{F}(y)| dm_n(y) < \infty$, *where* $|y| = (y_1^2 + \cdots + y_n^2)^{1/2}$.

Proof. Choose $\psi \in \mathcal{D}(\Omega)$ (resp. $\mathcal{D}_p(\Omega)$) so that $\psi = 1$ on $\bar{\omega}$, and define F on R^n by

$$F(x) \quad = \quad \begin{cases} \psi(x) f(x) & \text{if} \quad x \in \Omega \\ 0 \quad \text{if} \quad x \notin \Omega. \end{cases}$$

Then $F \in (L^1 \cap L^2)(R^n)$ and $f = F$ in ω. By the Plancherel Theorem [3], we have

$$\int_{R^n} |\hat{F}|^2 dm_n < \infty \quad \text{and} \int_{R^n} y_i^{2r} |\hat{F}(y)|^2 dm_n(y) < \infty \quad (1 \leq i \leq n).$$

Hence,

$$\int_{R^n}(1+|y|)^{2r}|\hat{F}(y)|dm_n < \infty$$

since $(1+|y|)^{2r} < (2n+2)^r(1+y_1^{2r}+\cdots y_n^{2r})$. By the Schwarz inequality we have

$$\begin{aligned}
&\{\int_{R^n}(1+|y|)^q|\hat{F}(y)|dm_n(y)\}^2 \\
&\leq \int_{R^n}(1+|y|)^{2r}|\hat{F}(y)|^2 dm_n(y)\cdot\int_{R^n}(1+|y|)^{2q-2r}dm_n(y) \\
&= M\sigma_n\int_0^\infty(1+t)^{2q-2r}t^{n-1}dt < \infty
\end{aligned}$$

where $M = \int_{R^n}(1+|y|)^{2r}|\hat{F}(y)|^2 dm_n(y)$ and σ_n is the $(n-1)$-dimensional volume of the unit sphere in R^n.

LEMMA 2.4. *(The Inversion Theorem, [3]) If $f \in L^1(R^n), \hat{f} \in L^1(R^n)$, and $f_0 = \int_{R^n}\hat{f}e_x dm_n \quad (x \in R^n)$, then $f(x) = f_0(x)$ for almost every $(x \in R^n)$.*

THEOREM 2.5. *(Sobolev's Lemma) Under the assumptions of the Lemma 2.3, there exists a function $f_0 \in C^{(q)}(\Omega)$ such that $f_0 = f(x)$ for almost every $x \in \Omega$.*

Proof. Let F be the function defined in Lemma 2.3. Dedine $F_\omega(x) = \int_{R^n}\hat{F}e_x dm_n \quad (x \in R^n)$. Then $F_\omega = F$ a.e. on R^n by the Inversion Theorem 2.4. If $x = (x_1, \cdots, x_n)$ and $x' = (x_1+\epsilon, x_2, \cdots, x_n), \epsilon \neq 0$, then

$$\frac{F_\omega(x') - F_\omega(x)}{i\epsilon} = \int_{R^n} y_1\hat{F}(y)\frac{e^{i\epsilon y_1}-1}{iy_1\epsilon}e^{iyx}dm_n(y).$$

The dominated convergence theorem can be applied, since $y_1\hat{F} \in L^1$, and yields

$$\frac{\partial}{\partial x_1}F_\omega(x) = i\int_{R^n} y_1\hat{F}(y)e^{iyx}dm_n(y).$$

Iteration of the proof of the above leads therefore to conclusion $F_\omega \in C^{(q)}(R^n)$. Consequently, $f = F_\omega$ a.e. in ω. Define $f_0 = F_\omega(x)$, if $x \in \omega$. Then the function f_0 is the desired one.

COROLLARY 2.6. *If all distribution (resp. generalized distribution) derivatives of f, Theorem 2.5, are locally L^2 in Ω, then $f_0 \in C^\infty(\Omega)$.*

Proof. By the above Theorem it is clear.

3. The spaces $\mathcal{D}_p(\Omega)$

We recall some properties of the spaces $\mathcal{D}_p(\Omega)$. If $p_1 \prec p_2$, then $\mathcal{D}_{p_1} \subset \mathcal{D}_{p_2}$ and $\mathcal{D}_{p_1}(\Omega)$ is dense in $\mathcal{D}_{p_2}(\Omega)$ for each open $\Omega \subset R^n$. Conversely, if for some compact $K \subset R^n$ with $\overset{\circ}{K} \neq \phi, \mathcal{D}_{p_1}(K) \subset \mathcal{D}_{p_2}(K)$, then $p_2 \prec p_1$. Let $p \in M_0(n)$. Then $\mathcal{D}_p(\Omega) \subset \mathcal{D}(\Omega)$ for every open Ω in R^n if and only if $p_0 \prec p$, where $p_0(x) = \ln(1+|x|) \quad (x \in R^n)$.

PROPOSITION 3.1. *The space $\mathcal{D}(\Omega)$ (resp $\mathcal{D}_p(\Omega)$) is the set of all functions ϕ on an open set $\Omega \subset R^n$, where all distribution (resp. generalized distribution) derivatives are locally L^2 in Ω, and each ϕ has compact support in Ω, with the limit topology.*

Proof. It is clear by Sobolev's Lemma.

PROPOSITION 3.2. *Let $p_0(x) = \ln(1 + |x|)$. Then the test function space $\mathcal{D}_{p_0}$ is the set of all functions ϕ in $L^1(R^n)$ with compact support such that*

$$||\hat{\phi}||_n^{(p_0)} = \int_{R^n} (1 + |t|)^n |\hat{\phi}(t)| dt < \infty$$

for all nonegative intergers n. Therefore, the space $\mathcal{D}_{p_0}$ coinsides with the space $\mathcal{D}$.

Proof. It is obvious by Lemma 2.3 and Theorem 2.5.

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