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ON RULED REAL HYPERSURFACES IN A COMPLEX SPACE FORM II

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1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0, c = 0$ or $c < 0$.

The classification and the structure of the homogeneous real hypersurfaces in $M_n(c)$ are investigated by many authors. See Takagi [9], Berndt [2] and etc.

As an example of special real hypersurfaces of $P_n\mathbb{C}$, we can give a ruled real hypersurface. Let $\gamma : I \to M_n(c)$ be any regular curve. For any $t(\in I)$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in M^{(t)}_{n-1}(c) : t \in I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$, which is called a *ruled real hypersurface*. In [4,5], Kimura obtained some properties about a ruled real hypersurface M of $P_n\mathbb{C}$.

Let M be a real hypersurface of $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure of $M_n(c)$. Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution*. And the second fundamental form is said to be *η-parallel* if the shape operator A satisfies $g(\nabla_X A(Y), Z) = 0$ for any vector fields X, Y and Z in T_0, where $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X. Then Kimura and Maeda [6] and Ahn, Lee and Suh [1] proved the following

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THEOREM A. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 3$. Then the second fundamental form is η-parallel and the holomorphic distribution T_0 is integrable if and only if M is locally a ruled real hypersurface.*

THEOREM B. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. Assume that ξ is not principal. Then it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{1.1}$$

for any vector fields X and Y in T_0 and the second fundamental form is η-parallel if and only if M is locally a ruled real hypersurface.

Now, let S be the Ricci tensor of M. Then S is said to be *η-parallel* if $g(\nabla_X S(Y), Z) = 0$ for any vector fields X, Y and Z in T_0. Even though the second fundamental form for the ruled real hypersurfaces is η-parallel, the Ricci tensor is not necessarilly η-parallel. In fact, if we put $A\xi = \alpha\xi + \beta U$ for a unit vector field U in T_0 and smooth functions α and β on M, then the covariant derivative ∇A of the shape operator A is given by (see [8])

$$\nabla_X A(Y) = f(X, Y)\xi, \quad X, Y \in T_0,$$

where we put

$$f(X, Y) = \beta^2\{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\} - \frac{c}{4}g(\phi X, Y).$$

This means that A is η-parallel. Furthermore, the covariant derivative ∇S of the Ricci tensor S satisfies

$$g(\nabla_X S(Y), Z) = -\beta\{g(Y, U)f(X, Z) + g(Z, U)f(X, Y)\} \tag{1.2}$$

for any vector fields X, Y and Z in T_0.

The purpose of this article is to prove the following characterization of ruled real hypersurfaces in terms of the Ricci tensor.

THEOREM. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies (1.1) and (1.2) and if the structure vector field ξ is not principal, then M is locally a ruled real hypersurface.*

2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $(M_n(c), \bar{g})$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor $\bar{g}$ on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties :

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$\text{(2.1)} \qquad \nabla_X \xi = \phi AX, \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained:

$$\text{(2.2)} \qquad \begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$\text{(2.3)} \quad \nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Next, we assume that it satisfies

$$g((A\phi - \phi A)X, Y) = 0 \tag{2.4}$$

for any vector fields X and Y in T_0. Let $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in T_0, and α and β are smooth functions on M. Then we have

$$\begin{aligned} &g(\nabla_X A(Y), \phi Z) + g(\nabla_X A(Z), \phi Y) \\ &\quad = \beta\{g(Y,U)g(AX,Z) + g(Z,U)g(AX,Y) \\ &\qquad - g(Y,\phi U)g(\phi AX, Z) - g(Z,\phi U)g(\phi AX, Y)\} \end{aligned} \tag{2.5}$$

for any vector fields X, Y and Z in T_0. Furthermore, (2.4) implies

$$(A\phi - \phi A)X = -\beta g(X, \phi U)\xi \tag{2.6}$$

for any vector field X in T_0. Making use of this property, we have

$$g(\nabla_X A(Y), Z) = \beta \mathfrak{S}\, g(AX, Y)g(Z, \phi U) \tag{2.7}$$

for any vector fields X, Y and Z in T_0, where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z, which is proved by Ahn, Lee and Suh [1].

Now, we here calculate the covariant derivative of the Ricci tensor S. Since the Ricci tensor S is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2$$

for the identity transformation I and the trace h of A, we get

$$\begin{aligned} \nabla_X S(Y) = &-\frac{3c}{4} g(\phi AX, Y)\xi + dh(X)AY \\ &+ h\nabla_X A(Y) - \nabla_X A(AY) - A\nabla_X A(Y), \end{aligned}$$

from which it turns out to be

$$\begin{aligned} g(\nabla_X S(Y), Z) =& dh(X)g(AY, Z) + hg(\nabla_X A(Y), Z) \\ &- g(\nabla_X A(Y), AZ) - g(\nabla_X A(Z), AY) \end{aligned} \tag{2.8}$$

for any vector fields X, Y and Z in T_0. Accordingly, we have

$$(2.9)\quad \begin{aligned} &g(\nabla_X S(Y), \phi Z) + g(\nabla_X S(Z), \phi Y) \\ &\quad = h\{g(\nabla_X A(Y), \phi Z) + g(\nabla_X A(Z), \phi Y)\} \\ &\quad\quad - g(\nabla_X A(Y), A\phi Z) - g(\nabla_X A(\phi Y), AZ) \\ &\quad\quad - g(\nabla_X A(Z), A\phi Y) - g(\nabla_X A(\phi Z), AY) \end{aligned}$$

for any vector fields X, Y and Z in T_0, where we have used the assumption (2.4). Since $A\phi Z = \phi AZ - \beta g(Z, \phi U)\xi$ for any vector field Z in T_0 by (2.6), we have

$$\begin{aligned} &g(\nabla_X A(Y), A\phi Z) + g(\nabla_X A(\phi Y), AZ) \\ &\quad = g(\nabla_X A(Y), \phi AZ) + g(\nabla_X A((AZ)_0), \phi Y) \\ &\quad\quad + \beta\{g(Z, U)g(\nabla_X A(\phi Y), \xi) - g(Z, \phi U)g(\nabla_X A(Y), \xi)\} \end{aligned}$$

for any vector fields X, Y and Z in T_0, where we denote by $(AZ)_0$ the T_0-component of the vector field AZ. By using (2.5), the above equation is reformed as

$$\begin{aligned} &g(\nabla_X A(Y), A\phi Z) + g(\nabla_X A(\phi Y), AZ) \\ &\quad = \beta\{g(Y, U)g(AX, AZ) + g(AZ, U)g(AX, Y) \\ &\quad\quad - g(Y, \phi U)g(\phi AX, AZ) - g(AZ, \phi U)g(\phi AX, Y) \\ &\quad\quad - \beta^2 g(X, U)g(Y, U)g(Z, U) + g(Z, U)g(\nabla_X A(\phi Y), \xi) \\ &\quad\quad - g(Z, \phi U)g(\nabla_X A(Y), \xi)\}. \end{aligned}$$

From (2.5), (2.9) and the above equation, we obtain

$$(2.10)\quad \begin{aligned} &g(\nabla_X S(Y), \phi Z) + g(\nabla_X S(Z), \phi Y) \\ &\quad = \beta[h\{g(Y, U)g(AX, Z) + g(Z, U)g(AX, Y) \\ &\quad\quad\quad - g(Y, \phi U)g(\phi AX, Z) - g(Z, \phi U)g(\phi AX, Y)\} \\ &\quad\quad - g(Y, U)g(AX, AZ) - g(AZ, U)g(AX, Y) \\ &\quad\quad + g(Y, \phi U)g(\phi AX, AZ) + g(AZ, \phi U)g(\phi AX, Y) \\ &\quad\quad - g(Z, U)g(AX, AY) - g(AY, U)g(AX, Z) \\ &\quad\quad + g(Z, \phi U)g(\phi AX, AY) + g(AY, \phi U)g(\phi AX, Z) \\ &\quad\quad - g(Y, U)g(\nabla_X A(\phi Z), \xi) - g(Z, U)g(\nabla_X A(\phi Y), \xi) \\ &\quad\quad + g(Y, \phi U)g(\nabla_X A(Z), \xi) + g(Z, \phi U)g(\nabla_X A(Y), \xi) \\ &\quad\quad + 2\beta^2 g(X, U)g(Y, U)g(Z, U)] \end{aligned}$$

for any vector fields X, Y and Z in T_0.

Next, taking account of the first equation of (2.1), we have

$$\begin{aligned} g(\nabla_X A(Y), \xi) = &\alpha g(\phi AX, Y) - g(\phi AX, AY) \\ &+ d\beta(X) g(Y, U) + \beta g(\nabla_X U, Y), \end{aligned} \tag{2.11}$$

and hence, by the property of the structure tensor ϕ, we get also

$$\begin{aligned} g(\nabla_X A(\phi Y), \xi) = \alpha g(AX, Y) - g(AX, AY) + \beta^2 g(X, U) g(Y, U) \\ - d\beta(X) g(Y, \phi U) + \beta g(\nabla_X U, \phi Y) \end{aligned}$$

for any vector fields X and Y in T_0. By substituting the above two equations into (2.10) and by the straightforward calculation, this relation is reformed as follows ;

$$\begin{aligned} &g(\nabla_X S(Y), \phi Z) + g(\nabla_X S(Z), \phi Y) \\ &\quad = \beta[(h - \alpha)\{g(Y, U) g(AX, Z) + g(Z, U) g(AX, Y) \\ &\qquad - g(Y, \phi U) g(\phi AX, Z) - g(Z, \phi U) g(\phi AX, Y)\} \\ &\qquad - g(AY, U) g(AX, Z) - g(AZ, U) g(AX, Y) \\ &\qquad + g(AY, \phi U) g(\phi AX, Z) + g(AZ, \phi U) g(\phi AX, Y) \\ &\qquad + 2 d\beta(X) \{g(Y, U) g(Z, \phi U) + g(Z, U) g(Y, \phi U)\} \\ &\qquad - \beta\{g(Y, U) g(\nabla_X U, \phi Z) + g(Z, U) g(\nabla_X U, \phi Y) \\ &\qquad - g(Y, \phi U) g(\nabla_X U, Z) - g(Z, \phi U) g(\nabla_X U, Y)\}] \end{aligned} \tag{2.12}$$

for any vector fields X, Y and Z in T_0.

Last, we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [3] and [7] that α is constant on M and it satisfies

$$A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(A\phi + \phi A). \tag{2.13}$$

3. Proof of Theorem

In this section, we shall consider a characterization of ruled real hypersurfaces in terms of the Ricci tensor S. Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. Let us first assume that the structure vector field ξ is not principal. So, we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. By the assumption, the function β does not vanish identically on M. And we also assume the following conditions:

$$g((A\phi - \phi A)X, Y) = 0, \tag{1.1}$$

$$g(\nabla_X S(Y), Z) = -\beta\{g(Y,U)f(X,Z) + g(Z,U)f(X,Y)\} \tag{1.2}$$

for any vector fields X, Y and Z in T_0, where

$$f(X,Y) = \beta^2\{g(X,U)g(Y,\phi U) + g(X,\phi U)g(Y,U)\} - \frac{c}{4}g(\phi X, Y).$$

Under the assumption (1.2), it follows from (2.12) that we have

$$\begin{aligned}
\beta[(h-\alpha)\{&g(Y,U)g(AX,Z) + g(Z,U)g(AX,Y) \\
&- g(Y,\phi U)g(\phi AX,Z) - g(Z,\phi U)g(\phi AX,Y)\} \\
&-g(AY,U)g(AX,Z) - g(AZ,U)g(AX,Y) \\
&+g(AY,\phi U)g(\phi AX,Z) + g(AZ,\phi U)g(\phi AX,Y) \\
&+2d\beta(X)\{g(Y,U)g(Z,\phi U) + g(Z,U)g(Y,\phi U)\} \\
&-\beta\{g(Y,U)g(\nabla_X U,\phi Z) + g(Z,U)g(\nabla_X U,\phi Y) \\
&\quad - g(Y,\phi U)g(\nabla_X U,Z) - g(Z,\phi U)g(\nabla_X U,Y)\} \\
&+g(Y,U)f(X,\phi Z) + g(Z,U)f(X,\phi Y) \\
&-g(Y,\phi U)f(X,Z) - g(Z,\phi U)f(X,Y)] \\
=&\ 0
\end{aligned} \tag{3.1}$$

for any vector fields X, Y and Z in T_0. Putting $Y = Z = U$ in this equation, we get

$$\begin{aligned}
&\beta^2 g(\nabla_X U, \phi U) \\
&\quad = \beta\left\{(h-\alpha-\gamma)g(AX,U) + \left(\beta^2 - \frac{c}{4}\right)g(X,U)\right\}
\end{aligned} \tag{3.2}$$

for any vector field X in T_0, where γ is the function defined by $g(AU, U)$. Again, putting $Y = U$ and $Z = \phi U$ in (3.1), we see

$$(3.3)\quad \beta d\beta(X) = -\beta\left\{(h-\alpha-\gamma)g(AX,\phi U) - \left(\beta^2+\frac{c}{4}\right)g(X,\phi U)\right\}$$

for any vector field X in T_0. Let T_1 be a distribution defined by the subspace $T_1(x) = \{u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0\}$. We consider about any vector fields X in T_0 and $Y = Z$ in T_1 in (3.1), and we then get

$$\beta\{g(AY,U)g(AX,Y) - g(AY,\phi U)g(\phi AX,Y)\} = 0.$$

Accordingly, we have

$$(3.4)\qquad \beta\{g(AY,\phi U)A\phi Y + g(AY,U)AY\} = 0, \quad Y \in T_1.$$

Now, let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x) \neq 0$. We here prove the following

LEMMA 3.1. *The distribution T_1 is A-invariant on M_0.*

Proof. We can put $AU = \beta\xi + \gamma U + \delta U_1$ and $A\phi U = \gamma\phi U + \delta\phi U_1$, where U_1 is a unit vector field in T_1, and γ, δ and ε are smooth functions on M_0. Let M_1 be an open subset of M_0 defined by $M_1 = \{x \in M_0 : \delta(x) \neq 0\}$. Suppose that M_1 is not empty. Then we have by (3.4)

$$g(Y,U_1)AY + g(Y,\phi U_1)A\phi Y = 0, \quad Y \in T_1$$

on M_1. Putting $Y = U_1$ in this equation, $AU_1 = 0$ and hence $A\phi U_1 = 0$ by (1.1). Furthermore, we get the following equation

$$g(Y,U_1)AZ + g(Z,U_1)AY + g(Y,\phi U_1)A\phi Z + g(Z,\phi U_1)A\phi Y = 0,$$

for any vector fields Y and Z in T_1. Putting $Y = U_1$ in the above equation, we have $AZ = 0$ for any vector field Z in T_1. Thus T_1 is A-invariant on M_1 and hence $L(\xi, U, \phi U)$ is also A-invariant on M_1, where $L(\xi, U, \phi U)$ is a distribution defined by the subspace $L_x(\xi, U, \phi U)$ of the tangent space T_xM spanned by the tangent vectors $\xi(x), U(x)$ and

$\phi U(x)$ at any point x in M_1. Therefore $\delta = 0$ on M_1, a contradiction. It completes the proof. □

Consequently, we get

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi + \gamma U, \\ A\phi U = \gamma\phi U \end{cases}$$

on M_0. Hence we have by (3.1)

$$\begin{aligned}(3.5)\qquad &(h-\alpha-\gamma)\{g(Y,U)g(AX,Z)+g(Z,U)g(AX,Y)\\ &\qquad -g(Y,\phi U)g(\phi AX,Z)-g(Z,\phi U)g(\phi AX,Y)\}\\ &\quad +2d\beta(X)\{g(Y,U)g(Z,\phi U)+g(Z,U)g(Y,\phi U)\}\\ &\quad -\beta\{g(Y,U)g(\nabla_X U,\phi Z)+g(Z,U)g(\nabla_X U,\phi Y)\\ &\qquad -g(Y,\phi U)g(\nabla_X U,Z)-g(Z,\phi U)g(\nabla_X U,Y)\}\\ &\quad +g(Y,U)f(X,\phi Z)+g(Z,U)f(X,\phi Y)\\ &\quad -g(Y,\phi U)f(X,Z)-g(Z,\phi U)f(X,Y)]\\ &\quad =0\end{aligned}$$

for any vector fields X, Y and Z in T_0. Putting $Y = U$ and taking Z in T_1 in (3.5), we obtain

$$\beta g(\nabla_X U,\phi Z) = (h-\alpha-\gamma)g(AX,Z) - \frac{c}{4}g(X,Z).$$

Accordingly, we have by (3.2)

$$\begin{aligned}(3.6)\qquad \beta\nabla_X U &= (h-\alpha-\gamma)\phi AX - \frac{c}{4}\phi X + \beta\gamma g(X,\phi U)\xi\\ &\quad + \left\{\gamma(h-\alpha-\gamma)-\frac{c}{4}\right\}g(X,\phi U)U + \beta^2 g(X,U)\phi U\end{aligned}$$

for any vector field X in T_0.

LEMMA 3.2. *$\gamma = 0$ on M_0.*

Proof. Let M_2 be the open subset of M_0 consisting of points x at which $\gamma(x) \neq 0$. Suppose that M_2 is not empty. The discussion is considered on the subset M_2. Substituting (3.3) and (3.6) into (2.11), we get

$$(3.7)\qquad \begin{aligned} g(\nabla_X A(Y),\xi) &= -g(A\phi AX,Y)+(h-\gamma)g(\phi AX,Y)\\ &-\frac{c}{4}g(\phi X,Y)+\beta^2\{g(X,U)g(Y,\phi U)+g(X,\phi U)g(Y,U)\}\end{aligned}$$

for any vector fields X and Y in T_0. Interchanging X and Y in the above equation and applying (2.3), we have

$$(3.8)\qquad A\phi AX-(h-\gamma)\phi AX=-\beta\gamma g(X,\phi U)\xi,\quad X\in T_0,$$

from which together with $A\phi U=\phi AU=\gamma\phi U$ it follows that

$$(3.9)\qquad h-2\gamma=0.$$

Accordingly, putting $Y=U$ in (3.7), we get

$$g(\nabla_X A(U),\xi)=\left(\beta^2+\frac{c}{4}\right)g(X,\phi U),\quad X\in T_0.$$

By the assumption (1.2), we have

$$g(\nabla_X S(U),U)=-2\beta\left(\beta^2+\frac{c}{4}\right)g(X,\phi U),\quad X\in T_0.$$

And, by (2.8) and (3.9), we can get

$$g(\nabla_X S(U),U)=\gamma dh(X)-2\beta g(\nabla_X A(U),\xi),\quad X\in T_0.$$

Hence we obtain by the above three equations

$$(3.10)\qquad dh(X)=0,\quad X\in T_0.$$

From (3.9) and (3.10), we get $d\gamma(X)=0$. Thus we have

$$\nabla_X A(U)=d\beta(X)\xi+\beta\phi AX+\gamma\nabla_X U-A\nabla_X U,\quad X\in T_0.$$

On the other hand, we get by (2.7)

$$g(\nabla_X A(U),Y)=\beta\gamma\{g(X,U)g(Y,\phi U)+g(X,\phi U)g(Y,U)\},$$

from which together with (3.6), (3.8) and the above equation it follows that

$$(3.11)\qquad \left(\beta^2+\frac{c}{4}\right)A\phi X-\frac{c}{4}\gamma\phi X-\beta^2\gamma\{g(X,U)\phi U+2g(X,\phi U)U\} \equiv 0 \pmod{\xi},\quad X\in T_0.$$

Since T_1 is A-invariant, let $X \in T_1$ be a principal vector field corresponding to the principal curvature λ. By (3.8) and (3.11), we have

$$\lambda^2-\gamma\lambda=0,\qquad \left(\beta^2+\frac{c}{4}\right)\lambda-\frac{c}{4}=0.$$

This means that $\gamma = 0$ on M_2, a contradiction. It concludes the proof. □

Consequently, we have

$$\begin{cases} A\xi=\alpha\xi+\beta U,\\ AU=\beta\xi,\\ A\phi U=0\end{cases}$$

on M_0.

Next, we shall prove the following

LEMMA 3.3. *$AX = 0$ for any vector field X in T_1 on M_0.*

Proof. Under the property $\gamma = 0$, we see

$$(3.3')\qquad d\beta(X)=\left(\beta^2+\frac{c}{4}\right)g(X,\phi U),$$

and

$$(3.8')\qquad A\phi AX-h\phi AX=0$$

for any vector field X in T_0. Let $X \in T_1$ be a principal vector field corresponding to the principal curvature λ. We have by (3.8')

$$\lambda^2-h\lambda=0.$$

So, $\lambda = 0$ or $\lambda = h$. Suppose that there is a principal curvature $\lambda = h(\neq 0)$.. Then we obtain by (3.6) and (3.8')

$$\beta \nabla_X A(U) = \left(-h^2 + h\alpha + \beta^2 + \frac{c}{4}\right) \phi AX, \quad X \in T_1.$$

Since $g(\nabla_X A(U), Y) = 0$ for any vector fields X and Y in T_0 by (2.7), we get

$$h^2 - h\alpha - \beta^2 - \frac{c}{4} = 0. \tag{3.12}$$

For any fixed point x in M_0, let V be the eigenspace at point x corresponding to the eigenvalue $\lambda = h(\neq 0)$. We set dim $V = 2p > 0$. Then $\alpha = (1 - 2p)h$. Consequently, we have by (3.12)

$$2ph^2 = \beta^2 + \frac{c}{4}.$$

Hence we get by (3.3')

$$dh(X) = h\beta g(X, \phi U) \tag{3.13}$$

for any vector field X in T_0. As is well known, the Ricci formula for the shape operator A is given by

$$\nabla_X \nabla_Y A(Z) - \nabla_Y \nabla_X A(Z) = R(X, Y)(AZ) - A(R(X, Y)Z)$$

for any vector fields X, Y and Z. Let Y_0 be a unit vector field in T_1 such that $AY_0 = hY_0$. Putting $X = \phi U$ and $Y = Z = Y_0$ in the Ricci formula, we can obtain $c = 0$ by (2.2), (2.7), (3.3'), (3.13) and Lemma 3.2, a contradiction. This means that $AX = 0$ for any vector field X in T_1. □

Proof of Theorem. Suppose that the interior $Int(M - M_0)$ of $M - M_0$ is not empty. On the subset, the function β is vanishes identically and therefore ξ is principal. Thus we have

$$(A\phi - \phi A)\xi = 0.$$

For any principal vector field X in T_0 with principal curvature λ, the condition (1.1) is reduced to $A\phi X = \lambda \phi X + \theta(X)\xi$, where θ is a 1-form on $Int(M - M_0)$. From $A\xi = \alpha\xi$, the inner product of $A\phi X$ and ξ gives us to $\theta(X) = 0$. This means that

$$A\phi - \phi A = 0 \tag{3.14}$$

on $Int(M - M_0)$. Since ξ is principal on $Int(M - M_0)$, we have by (2.13)

$$(2\lambda - \alpha)A\phi X = \left(\frac{c}{2} + \alpha\lambda\right)\phi X.$$

Using (3.14) and the above equation, we get

$$2\lambda^2 - 2\alpha\lambda - \frac{c}{2} = 0, \tag{3.15}$$

from which it follows that all principal curvatures are non-zero constant on $Int(M - M_0)$. Since we assume that the set M_0 is not empty, (3.14) means that

$$g(AX, Y) = 0, \quad X, Y \in T_0$$

on M_0. So, it follows from this and (3.1) that we get

$$AX = g(AX, \xi)\xi = \beta g(X, U)\xi$$

for any vector field X in T_0. Hence, by Lemma 3.2 and Lemma 3.3, we have

$$AU = \beta\xi, \quad AX = 0 \tag{3.16}$$

for any vector field X in T_0 orthogonal to U. By means of the continuity of principal curvarures, (3.15) and (3.16) lead a contradiction. It shows that $Int(M - M_0)$ must be empty. Thus the open set M_0 is a dense subset of M. By the continuity of principal curvatures again, we see that the shape operator satisfies the condition (3 16) on the whole M. Therefore the distribution T_0 is integrable on M. Moreover the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vector fields are ξ and C. Since we have

$$\bar{g}(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = 0$$

and

$$\bar{g}(\bar{\nabla}_X Y, C) = g(AX, Y) = 0$$

for any vector fields X and Y in T_0 by (2.1) and (3.16), where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$, it is seen that the submanifold is totally geodesic in $M_n(c)$. Since T_0 is also J-invariant, its integral manifold is a complex manifold and hence it is a complex space form $M_{n-1}(c)$. Thus M is locally a ruled real hypersurface. □

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