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A NOTE ON SINGULAR COMPACTIFICATIONS

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Throughout this paper, all topological spaces concerned are assumed to be Hausdorff and the space X to be noncompact and locally compact.

For Hausdorff compactifications αX and γX of X, we say that $\gamma X \leq \alpha X$ if there is a continuous map $f : \alpha X \to \gamma X$ such that $f \circ \alpha = \gamma$. Then, the family of all Hausdorff compactifications of X is a complete lattice with this partial order $\leq$. Let Y be compact and let $f : X \to Y$ be continuous with $f(X)$ dense in Y. The subset $S(f)$ of Y defined by $\{p \in Y|$ for any neighborhood U of p, the closure of $f^{-1}(U)$ in X is not compact $\}$ is called the singular set of f. And also, f is called singular([1],[2]) if $S(f) = Y$. The singular set $S(f)$ is equal to the set $L(f) = \cap\{Cl_Y f(X - F)|F$ is compact in $X\}$([3]) and $S(f) = L(f)$ is a remainder of X([7]).

For a singular map $f : X \to Y$, the singular compactification of X induced by f, which is denoted by $X \cup_f S(f)$, is constructed as follows([6],[8]);

On the set $X \cup S(f)$, basic neighborhoods of $p \in X$ are the same in X and $p \in S(f) = Y$ has basic neighborhoods of the form $V \cup \{f^{-1}(V) - F\}$, where V is a neighborhood of p and F is any compact subset in X.

This is a generalization of the double circumference construction of Alexandroff and Urysohn([6]). Let $C^*(X)$ be the set of all continuous and bounded map from X to the real line R For a compactification αX of X and f in $C^*(X)$, we denote f^α the extension of f to αX if exists. Let $C_\alpha(X)$ denote the set of f in $C^*(X)$ which have extension to αX, and $S^\alpha(S^*)$ denote the set of f in $C_\alpha(X)(C^*(X))$ which is singular. In this note, we will show that for a connected space X, X has no 2-point compactification if and only if $\alpha X = \sup\{X \cup_f S(f)|f \in S^\alpha\}$ for any compactification αX of X, and that if X is weakly 1-complemented, then $\alpha X = \sup\{X \cup_f S(f)|f \in S^\alpha\}$ for any compactification αX of X.

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LEMMA 1([5]). *If f is in $C_\alpha(X)$, then $f^\alpha(\alpha X - X) = S(f)$.*

R.E.Chandler and G.D.Faulkner obtained a necessary and sufficient condition for a compactification αX to be $\sup\{X \cup_f S(f) | f \in \mathcal{G}\}$, which is useful.

PROPOSITION 2([5]). *Let αX be a compactification of X, and let $\mathcal{G}$ be a subcollection of S^α. Then, $\alpha X = \sup\{X \cup_f S(f) | f \in \mathcal{G}\}$ if and only if $\mathcal{G}^\alpha = \{f^\alpha | f \in S^\alpha\}$ separates points in $\alpha X - X$.*

LEMMA 3. *If X has no 2-point compactification, then $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

Proof. The argument is similar to that of Theorem 3 of [5]. Suppose that X has no 2-point compactification and let αX be any compactification of X. Since X has no 2-point compactification, $\alpha X - X$ is connected by Lemma 6.16 of [4]. Let p and q be distinct points of $\alpha X - X$. Then, there exists a continuous map $f : \alpha X \to [0,1]$ such that $f(p) = 0$ and $f(q) = 1$. Let g be the restriction of f to X. Then, since $S(g) = f(\alpha X - X) = [0,1]$, we have that g is singular with the extension f to αX which separates p and q. Hence, by Proposition 2, we see that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$.

LEMMA 4([4],[9]). *X has n-point compactifications if and only if there exist n open, nonempty pairwise disjoint subsets $\{G_i\}_{i=1}^n$ of X such that $K = X - \cup_{i=1}^n G_i$ is compact but for each i, $K \cup G_i$ is not compact.*

DEFINITION 5. *A space X is called weakly 1-complemented if for any compact subset K of X, there exist a compact subset F and a connected subset C of X such that $K \subset F, K \cap C = \emptyset$ and $F \cup C = X$.*

PROPOSITION 6. *If X is weakly 1-complemented, then $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

Proof. By Lemma 3, it is sufficient to show that X has no 2-point compactification. If X has a 2-point compactification, then by Lemma 4, there exist open, nonempty pairwise disjoint subsets G_1 and G_2 of X such that $K = X - (G_1 \cup G_2)$ is compact but for each $i = 1, 2$, $K \cup G_i$ is not compact. Since X is weakly 1-complemented, there exist a compact subset F and a connected subset C of X such that $K \subset F, K \cap C = \emptyset$ and $F \cup C = X$. Then, we have that $C \subset G_i$ for some i. we may assume that $C \subset G_1$. Then, since $K \cup G_2$ is a closed subset of the compact Hausdorff space F, we have a contradiction that $K \cup G_2$ is compact.

We call a space X to be 1-complemented (or connected at infinity) if each compact subset K is contained in some compact subset F with $X - F$ connected. It is trivial that if X is 1-complemented, then it is weakly 1-complemented. So, we have the following as a Corollary.

COROLLARY 7. *If X is 1-complemented, then $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

LEMMA 8([5]). *If f is in S^α, then $X \cup_f S(f) \leq \alpha X$.*

PROPOSITION 9. *Let X be a connected space. Then, the following statements are equivalent.*

(1) *X has no 2-point compactification*
(2) *$\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

Proof. It is sufficient to prove that (2)⇒(1). Suppose that there exists 2-point compactification αX of X with $\alpha X - X = \{-\infty, +\infty\}$. We will show that $\alpha X \neq \sup\{X \cup_f S(f) | f \in S^\alpha\}$. If not, then by Lemma 8 $\alpha X = X \cup_f S(f)$ for some $f \in S^\alpha$ or $X \cup_f S(f) < \alpha X$ for any $f \in S^\alpha$. In the latter case, it is impossible that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ since the compactification which is strictly less than αX is unique 1-point compactification. In the former case, we have the contradiction that $\{-\infty, +\infty\} = S(f) = Cl_R(f(X))$ is connected. This complets the proof.

The above Proposition 9 doesn't hold if the connectedness of X is deleted as you see in the following Example.

EXAMPLE 10. *Let $X = (-\infty, 0] \cup [1, +\infty)$ in the real line R. Then, it is not difficult to show that X has no 3-point compactification using Lemma 4. So, by Lemma 6.12 of [4], we have that X has unique 2-point compactification. Next, we will show that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X. Let αX be a compactification of X and let p and q be distinct points in $\alpha X - X$.*

Case 1. p and q are in the same component U of $\alpha X - X$; Since αX is compact Hausdorff(so, normal), there exists a continuous map

$f : \alpha X \to [0,1]$ *such that* $f(p) = 0$ *and* $f(q) = 1$. *Let* g *be a restriction of* f *to* X. *Then,* $g(X)$ *is dense in* $[0,1]$ *since* $[0,1] = f(U) \subset f(\alpha X) = f(Cl_{\alpha X}(X)) \subset Cl_R(f(X)) = Cl_R(g(X)) \subset [0,1]$. *And also, since* $[0,1] = f(U) \subset f(\alpha X - X) = S(g) \subset [0,1]$, *we have that* g *is singular map such that its extension* f *separates* p *and* q.

Case 2. p *and* q *are in distinct components of* $\alpha X - X$; *Let* U *be the component of* p *and* γX *be the quotient space* $\alpha X/\{U, \alpha X - X - U\}$. *Then,* γX *is a 2-point compactification. Since the 2-point compactification of* X *is unique, we have that there exists a continuous map* $f : \alpha X \to [-\infty, 0] \cup [1, +\infty]$ *such that* $f(\alpha X - X) = \{-\infty, +\infty\}$, $f(x) = x$ *for* $x \in X$ *and* f *separates* p *and* q. *Define a continuous map* $h : [-\infty, 0] \cup [1, +\infty] \to \{0, 1\}$ *by* $h([-\infty, 0]) = 0$ *and* $h([1, +\infty]) = 1$, *and let* g *be the restriction of* $h \circ f$ *to* X. *Then,* g *is a singular map with the extension* $h \circ f$ *to* αX, *which separates* p *and* q. *Hence, by Proposition 2, we have that* $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$.

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