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OPERATIONS ON THE SET OF NATURAL NUMBERS BY THE RECURSION THEOREM

WON HUH

1. Introduction The purpose of this note is to explore the operations of addition and multiplication of the set ω of natural numbers as applications of the recursion theorem.

2. Preliminaries and Notations We shall assume the Bernays-Gödel -von Neumann axiomatics for Set Theory.

Since the existence of a successor set is assumed, a natural number is, by definition, an element of the minimal successor set ω. The immediate successor of an element n of ω is denoted by the symbol n^+, and the immediate predecessor of a non-zero element n of ω is denoted by n^-.

THEOREM 2.1 *Let R_{n^+} be a mapping of a set E into E for each $n \in \omega$. Then for each $e \in E$, there exists one and only one mapping*

$$F_e : \omega \longrightarrow E$$

such that

(1) $F_e(0) = e$, and
(2) $F_e(n^+) = R_{n^+}(F_e(n))$ *for each* $n \in \omega$.

Proof. Let $\mathcal{A} = \{G \subset \omega \times E \mid (0,e) \in G \wedge ((n,x) \in G \longrightarrow (n^+, R_{n^+}(x)) \in G \forall n \in \omega\}$; then since $\omega \times E$ is in $\mathcal{A}$, $\mathcal{A} \neq \emptyset$. Since $(0,e) \in G$ for each $G \in \mathcal{A}$, $(0,e) \in \cap\mathcal{A}$, and since for each $G \in \mathcal{A}$ and each $(n,x) \in \omega \times E$, $(n,x) \in G$ implies $(n^+, R_{n^+}(x)) \in G$, we obtain that for each $(n,x) \in \omega \times E$, $(n,x) \in \cap\mathcal{A}$ implies $(n^+, R_{n^+}(x)) \in \cap\mathcal{A}$, and hence, $\cap\mathcal{A} \in \mathcal{A}$. We claim that

$$F_e = \cap\mathcal{A} : \omega \longrightarrow E$$

with satisfying

$$(0,e) \in F_e \wedge (F_e(n), F_e(n^+)) \in R_{n^+} \quad \text{for each } n \in \omega.$$

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It is easy to see that $F_e = \cap\mathcal{A} \subset G$ for each $G \in \mathcal{A}$. We now proceed the proof of our claim in steps.

(i) We are going to show that $dom(F_e) = \omega$ by induction. Since $(0, e) \in F_e$, $0 \in dom(F_e)$. Let $n \in dom(F_e)$; then we can find an $x \in E$ such that $(n, x) \in F_e$, so that $(n^+, R_{n^+}(x)) \in F_e$, and hence $n^+ \in dom(F_e)$, showing that $dom(F_e) = \omega$.

(ii) We are going to show that F_e is a function. To this end, let

$$S = \{n \in \omega \mid (n, x) \in F_e \wedge (n, y) \in F_e \longrightarrow x = y\}.$$

We wish to show that $S = \omega$. To show $0 \in S$, we argue by contradiction: Assume $0 \notin S$; then there would be a $d \in E$ such that $(0, d) \in F_e$ with $e \neq d$; in this case, $G = F_e \setminus \{(0, d)\}$ would be in $\mathcal{A}$; indeed, $(0, e) \in G$ and if $(n, x) \in G$ then $(n, x) \in F_e$, so that $(n^+, R_{n^+}(x)) \neq (0, d)$ for each $n \in \omega$, that is, $(n, x) \in G$ implies $(n^+, R_{n^+}(x)) \in G$ for each $n \in \omega$, and hence $F_e \subset G$, contradicting $G \subset F_e \wedge F_e \neq G$, establishing $0 \in S$. We wish to show that $n \in S$ implies $n^+ \in S$. To this end, assume there were an $n \in \omega$ such that $n \in S \wedge n^+ \notin S$; by noting that letting $N = \omega \setminus \{0\}$,

$$\{n \in N \mid \forall x \in E \forall y \in E : (n^-, x) \in F_e \longrightarrow \\ ((n, R_n(x)) \in F_e \wedge (n, y) \in F_e \longrightarrow R_n(x) = y)\}$$

is a subset of S, there would be an $n \in S$, an $x \in E$, and a $y \in E$ such that

$$(n, x) \in F_e \wedge (n^+, R_{n^+}(x)) \in F_e \wedge (n^+, y) \in F_e \wedge R_{n^+}(x) \neq y.$$

Let $G = F_e \setminus \{(n^+, y)\}$; then since $(0, e) \neq (n^+, y)$, we have $(0, e) \in G$. Let $(k, t) \in G$; then $(k, t) \in F_e$, and hence $(k^+, R_{k^+}(t)) \in F_e$. In this case, we wish to show that $(n^+, y) \neq (k^+, R_{k^+}(t))$. To this purpose, assume $(n^+, y) = (k^+, R_{k^+}(t))$; then we would have $n^+ = k^+ \wedge y = R_{k^+}(t)$, so that $k = n$, and hence, $(n, x) \in F_e \wedge (k, t) = (n, t)$, so $t = x$ because $n \in S$, and we would have $y = R_{n^+}(x)$, contradicting $y \neq R_{n^+}(x)$, showing that $(k, t) \in G$ implies $(k^+, R_{k^+}(t)) \in G$, so that $G \in \mathcal{A}$ and $F_e \subset G$, contradicting $G \subset F_e \wedge F_e \neq G$. Thus, we have seen that the assumption that there is an $n \in \omega$ such that $n \in S \wedge n^+ \notin S$ is false, establishing that $S = \omega$. Since for each $(n, x) \in \omega \times E$, $(n, x) \in F_e$ implies $(n^+, R_{n^+}(x)) \in F_e$, we have $F_e(n^+) = R_{n^+}(F_e(n))$ for each $n \in \omega$. Thus we have proved that $F_e = \cap\mathcal{A}$ is a mapping of ω into E such that (1) $F_e(0) = e$, and (2) $F_e(n^+) = R_{n^+}(F_e(n))$ for each $n \in \omega$. (iii) It remains to prove that there is at most one such mapping F_e. To this end, assume there were two distinct such mappings F_e and F^*; then there would be an $m \in \omega \setminus \{0\}$ such that $F_e(m) \neq F^*(m)$. Letting $S = \{m \in \omega \mid F_e(m) \neq F^*(m)\}$, there would be a first member $k(\neq 0)$ of S such that $F_e(k) \neq F^*(k)$ and $F_e(k^-) = F^*(k^-)$, since $R_k(F_e(k^-)) = R_k(F^*(k^-))$, we would have $F_e(k) = F^*(k)$, contradicting $F_e(k) \neq F^*(k)$. Thus, we have completed our proof. □

If $R_{n^+} = f$ for each $n \in \omega$ we have the following **Recursion theorem**

THEOREM 2.2. *Let f be a mapping of a set E into itself, then for each $e \in E$, there exists one and only one mapping*

$$F_e : \omega \longrightarrow E$$

such that

(1) $F_e(0) = e$, *and*
(2) $F_e(n^+) = f(F_e(n))$ *for each* $n \in \omega$.

For the sake of later use, we discuss the following

THEOREM 2.3. *Let a be any fixed member of a set E and let a mapping $f : E \times \omega \longrightarrow E$ be given. Then for each $m \in \omega$, there exists one and only one mapping*

$$F_m : \omega \longrightarrow E$$

such that

(1) $F_m(0) = a$, *and*
(2) $F_m(n^+) = f(F_m(n), m)$ *for each* $n \in \omega$.

Proof. Let

$$\mathcal{A} = \{G \subset \omega \times E \mid (0, a) \in G \wedge \forall (n, x) \in \omega \times E : (n, x) \in G \longrightarrow (n^+, f(x, m)) \in G\};$$

then since $\omega \times E \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$, and since $(0, a) \in G$ for each $G \in \mathcal{A}$, $(0, a) \in \cap\mathcal{A}$. Since for each $G \in \mathcal{A}$ and each $(n, x) \in \omega \times E$, $(n, x) \in G$ implies $(n^+, f(x, m)) \in G$, we obtain, for each $(n, x) \in \omega \times E$, $(n, x) \in \cap\mathcal{A}$ implies $(n^+, f(x, m)) \in \cap\mathcal{A}$, so that $\cap\mathcal{A} \in \mathcal{A}$. It is easy to see that for each $G \in \mathcal{A}$, $\cap\mathcal{A} \subset G$. We claim that $F_m : \omega \longrightarrow E$, and satisfies
(1) $F_m(0) = a$, and
(2) $F_m(n^+) = f(F_m(n), m)$ for each $n \in \omega$.

We now proceed the proof of our claim in steps. (i) We are going to show that $dom(F_m) = \omega$ by induction. Since $(0, a)$ is a member of F_m,

$0 \in dom(F_m)$. Let $n \in dom(F_m)$; then there exists an $x \in E$ such that $(n,x) \in F_m$, and hence, $(n^+, f(x,m)) \in F_m$, so that $n^+ \in dom(F_m)$, showing that $dom(F_m) = \omega$.

(ii) We are going to show that F_m is a function. That is, it is enough to show that for each $n \in \omega$ and each pair of members x and y of E, $(n,x) \in F_m \wedge (n,y) \in F_m$ implies $x = y$. To this end, let $S = \{n \in \omega \mid \forall x \in E \forall y \in E : (n,x) \in F_m \wedge (n,y) \in F_m \longrightarrow x = y\}$. We wish to show that $S = \omega$. Assume $0 \notin S$; then there would be an x of E such that $(0,x) \in F_m \wedge a \neq x$. Consider $G = F_m \setminus \{(0,x)\}$; then $(0,a) \in G$, and if $(n,t) \in G$ then $(n^+, f(t,m)) \in F_m$, and $(0,x) \neq (n^+, f(t,m))$, showing that $(n^+, f(t,m)) \in G$ whenever $(n,t) \in G$, and hence, $G \in \mathcal{A}$,so $F_m \subset G$, contradicting $G \subset F_m \wedge F_m \neq G$. Hence, the assumption $0 \notin S$ is false, therfore, we have $0 \in S$. We wish to show that for each $n \in \omega$, $n \in S$ implies $n^+ \in S$. We argue by contradiction. Assume there were an $n \in \omega$ such that $n \in S \wedge n^+ \notin S$, by noting that letting $N = \omega \setminus \{0\}$

$$\begin{aligned}\{n \in N \mid \forall x \in E \forall y \in E :&(n^-, x) \in F_m \wedge (n, f(x,m)) \in F_m \\ &\wedge (n,y) \in F_m \longrightarrow f(x,m) = y\}\end{aligned}$$

is a subset of S, there would be an $n \in S$ such that

$$(n,x) \in F_m \wedge (n^+, f(x,m)) \in F_m \wedge (n^+, y) \in F_m \wedge y \neq f(x,m).$$

Let $G = F_m \setminus \{(n^+, y)\}$;then $(0,a) \in G$, and if $(k,t) \in G$ then $(k,t) \in F_m$, so that $(k^+, f(t,m)) \in F_m$. We wish to show that $(k^+, f(t,m)) \neq (n^+, y)$. To this purpose, assume $(k^+, f(t,m)) = (n^+, y)$; then $k^+ = n^+ \wedge f(t,m) = y$, so that $k = n \wedge f(t,m) = y$, and hence, $(k,t) = (n,t)$, since $(n,x) \in F_m$, we have $x = t$ because $n \in S$, so that $f(t,m) = f(x,m) = y$, contradicting $f(x,m) \neq y$. Thus, we conclude that $(k^+, f(t,m)) \neq (n^+, y)$, showing that $(k,t) \in G$ implies $(k^+, f(t,m)) \in G$ so that $G \in \mathcal{A}$, and hence, $F_m \subset G$, contradicting $G \subset F_m \wedge G \neq F_m$, showing that the assumption that there exists an $n \in \omega$ such that $n \in S \wedge n^+ \notin S$ is false. From which it follows that $S = \omega$, showing that F_m is a function. Since for each $(n,x) \in \omega \times E$, $(n,x) \in F_m$ implies $(n^+ . f(x,m)) \in F_m$, we have $F_m(n^+) = f(F_m(n), m)$ for each $n \in \omega$.

(iii) It remains to prove that there is at most one such mapping F_m. To this purpose, assume there were two distinct such mappings F_m and G_m; then there would be a non-zero n of ω such that $F_m(n) \neq G_m(n)$. Letting $W = \{n \in \omega \mid F_m(n) \neq G_m(n)\}$, $W \subset \omega$, and there would be a first member

$k(\neq 0)$ of W such that $F_m(k) \neq G_m(k)$ and $F_m(k^-) = G_m(k^-)$, from which it follows that $F_m(k) = f(F_m(k^-), m) = f(G_m(k^-$.

$), m) = G_m(k)$, contradicting the choice of k. Thus, the assumption that there are two distinct such mapping is false. □

By an **evaluation** of ω^ω, we mean a mapping

$$\phi : \omega^\omega \times \omega \longrightarrow \omega$$

such that $\phi(f, n) = f(n)$ for each $f : \omega \longrightarrow \omega$ and each $n \in \omega$.

3. Addition Since $f \subset \omega \times \omega$ defined by

$$(m, n) \in f \quad \text{if and only if} \quad n = m^+$$

is a mapping of ω into itself. By the recursion theorem, for each $m \in \omega$, there exists a unique mapping

$$S_m : \omega \longrightarrow \omega$$

such that

(1) $S_m(0) = m$, and
(2) $S_m(n^+) = f(S_m(n)) = (S_m(n))^+$ for each $n \in \omega$.

Let $\mathcal{A} = \{S_m \mid m \in \omega\}$, let $\phi_\mathcal{A}$ be a restriction of the evaluation ϕ of ω^ω to $\mathcal{A} \times \omega$, let $k : \omega \longrightarrow \mathcal{A}$ be defined by $k(m) = S_m$ for each $m \in \omega$, and let $1 : \omega \longrightarrow \omega$ be the identity mapping; then we obtain a mapping diagram

$$\omega \times \omega \xrightarrow{k \times 1} \mathcal{A} \times \omega \xrightarrow{\phi_\mathcal{A}} \omega$$

such that

$$\phi_\mathcal{A} \circ (k \times 1)(m, n) = \phi_\mathcal{A}(S_m, n) = S_m(n)$$

for each $n \in \omega$.

Letting

$$\phi_\mathcal{A} \circ (k \times 1) = \alpha,$$

we have the following addition operation on ω

THEOREM 3.1. *There exists a unique mapping* **called the addition**

$$\alpha : \omega \times \omega \longrightarrow \omega$$

such that

(1) $\alpha(m, 0) = m$ *for each* $m \in \omega$, *and*
(2) $\alpha(m, n^+) = (\alpha(m, n))^+$ *for each* m *and each* n *of* ω.

As an immediate consequence, we have the following

COROLLARY.

(1) *For each* $m \in \omega$, $\alpha(0, m) = m$.
(2) *For each pair of* m *and* n *of* ω, $\alpha(m^+, n) = (\alpha(m, n))^+$.

THEOREM 3.2. *The addition* α *on* ω *is associative, that is, for all members* l, m, *and* n *of* ω,

$$\alpha(\alpha(l, m), n) = \alpha(l, \alpha(m, n)).$$

Proof. The proof goes by the mathematical induction on n. Let

$$S = \{n \in \omega \mid \forall l \in \omega \forall m \in \omega : \alpha(\alpha(l, m), n) = \alpha(l, \alpha(m, n));$$

then $S \subset \omega$. Since $\alpha(\alpha(l, m), 0) = \alpha(l, m) = \alpha(l, \alpha(m, 0))$, we have $0 \in S$. Let $n \in S$;then for each $l \in \omega$ and each $m \in \omega$,

$$\alpha(\alpha(l, m), n) = \alpha(l, \alpha(m, n)),$$

and hence,

$$\begin{aligned} \alpha(\alpha(l, m), n^+) &= (\alpha(\alpha(l, m), n))^+ = (\alpha(l, \alpha(m, n)))^+ \\ &= \alpha(l, (\alpha(m, n))^+) = \alpha(l, \alpha(m, n^+)), \end{aligned}$$

showing that $n \in S$ implies $n^+ \in S$. Thus, we have proved that for all members l, m, and n of ω, $\alpha(\alpha(l, m), n) = \alpha(l, \alpha(m, n))$. □

4. Multiplication Since the addition α on ω is defined as a mapping

$$\alpha : \omega \times \omega \longrightarrow \omega,$$

due to Theorem 2.3, for each $m \in \omega$, there exists a unique mapping

$$F_m : \omega \longrightarrow \omega$$

such that

(1) $F_m(0) = 0$, and
(2) $F_m(n^+) = \alpha(F_m(n), m)$ for each $n \in \omega$.

Letting $\mathcal{A} = \{F_m \mid m \in \omega\}$, letting $\phi_{\mathcal{A}}$ be a restriction of the evaluation ϕ of ω^ω to $\mathcal{A} \times \omega$, letting $k : \omega \longrightarrow \mathcal{A}$ be defined by $k(m) = F_m$ for each $m \in \omega$, and letting $1 : \omega \longrightarrow \omega$ be the identity mapping, we have a following mapping diagram such that

$$\omega \times \omega \xrightarrow{k \times 1} \mathcal{A} \times \omega \xrightarrow{\phi_{\mathcal{A}}} \omega$$

satisfying $\phi_{\mathcal{A}} \circ (k \times 1)(m, n) = \phi_{\mathcal{A}}(F_m, n) = F_m(n)$ for each pair of m and n of ω.

Putting $\phi_{\mathcal{A}} \circ (k \times 1) = \mu$, we have the following multiplication operation on ω,

THEOREM 4.1. *There exists a unique multiplication operation*

$$\mu : \omega \times \omega$$

such that

(1) $\mu(m, 0) = 0$ *for each* $m \in \omega$,*and*

(2) *for each* $m \in \omega$ *and each* $n \in \omega$,

$$\mu(m, n^+) = \alpha(\mu(m, n), m).$$

As an immediate consequence, we have the following

COROLLARY. *1. For each* $n \in \omega$,

$$\mu(0, n) = 0.$$

2. For each $n \in \omega$,

$$\mu(1, n) = n.$$

For the sake of convenience, we make the usual notation:

DEFINITION 1. For each $n \in \omega$,

$$n^+$$

is denoted as

$$n + 1,$$

that is,

$$n + 1 = n^+.$$

2. For each pair of m and n of ω,

$$\alpha(m,n)$$

is denoted as

$$m+n,$$

that is,

$$m+n=\alpha(m,n).$$

3. For each pair of m and n of ω,

$$\mu(m,n)$$

is denoted as

$$m\cdot n \quad \text{or} \quad mn,$$

that is,

$$m\cdot n=\mu(m,n) \quad \text{or} \quad mn=\mu(m,n).$$

THEOREM 4.2. *The multiplication operation on ω is associative, that is, for all l,m, and n of ω,*

$$(lm)n=l(mn).$$

Proof. The proof goes by the mathematical induction on n. Let

$$S=\{n\in\omega \mid \forall l\in\omega\forall m\in\omega:(lm)n=l(mn)\};$$

then $S\subset\omega$ and $0\in S$. Suppose that $n\in S$; then $(lm)n=l(mn)$, and hence,

$$\begin{aligned}(lm)(n+1)&=(lm)n+(lm)\\&=l(mn)+(lm)\\&=l((mn)+m)\\&=l(m(n+1)),\end{aligned}$$

so that $(n+1)\in S$, establishing $S=\omega$. $\square$

Now, we study the distributivity of multiplication over addition.

THEOREM 4.3. *For all members l, m, and n of ω,*

$$l(m+n) = (lm) + (ln)$$

Proof. The proof goes by the mathematical induction on n. Let

$$S = \{n \in \omega \mid \forall l \in \omega \forall m \in \omega : l(m+n) = (lm) + (ln)\};$$

then $S \subset \omega$, and since $l(m+0) = lm$ and $(lm) + (l0) = lm$, $0 \in S$. Suppose that $n \in S$; then for each l and each m of ω, $l(m+n) = (lm) + (ln)$, and hence, for each l and each m of ω,

$$\begin{aligned} l(m+(n+1)) &= l((m+n)+1) \\ &= l(m+n) + l \\ &= ((lm) + (ln)) + l \\ &= (lm) + ((ln) + l) \\ &= (lm) + (l(n+1)), \end{aligned}$$

so that $(n+1) \in S$, establishing $S = \omega$. □

References

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Department of Mathematics
Pusan National University
Pusan, 609–735 Korea