Pusan Kyŏngnam Math. J 10(1994), No. 2, pp 373-380

# SIMPLICITY OF $C^*$-CROSSED PRODUCTS AND STABLE RANK

SUN YOUNG JANG AND HAI GON JE

## 1. Introduction

A $C^*$-dynamical system is a triple $(A, G, \alpha)$, consisting of a $C^*$-algebra $A$, a locally compact group $G$, and a pointwise norm continuous homomorphism $\alpha$ of $G$ into the group $Aut(A)$ of $^*$-automorphisms of $A$. If a $C^*$-dynamical system $(A, G, \alpha)$ is given, we can construct two $C^*$-algebras from the $C^*$-dynamical system $(A, G, \alpha)$, one is the crossed product $A \times_\alpha G$ and the other is the reduced crossed product $A \times_{\alpha r} G$. In this paper we study the equivalence conditions between properties of $C^*$-dynamical system $(A, G, \alpha)$ and ideal structures of the corresponding $C^*$-crossed product $A \times_\alpha G$ and the reduced crossed product $A \times_{\alpha r} G$ and the topological stable rank of $C^*$-crossed products. The problems of simplicity of $C^*$-crossed products are the $C^*$-analogue of von Neumann's problem. In 1940, von Neumann [ 5 ] proved that the crossed product of a commutative von Neumann algebra with a discrete group acting freely and ergodically is a factor. Later a sufficient condition for a $W^*$-crossed product $M \times_\alpha G$ to be a factor with a discrete group was given by Nakamura and Takeda [ 4 ] as the outerness of every $\alpha_g$, $g \neq e$. In the case of $C^*$-algebras, there are some results about the ideal structure of $C^*$-crossed products. If the group $G$ is abelian, Kishimoto [ 2 ], Olesen and Pedersen [ 6 ] investigated the ideal structure of $A \times_\alpha G$ by using concepts of the Connes' spectrum. And Kawamura and Tomiyama [ 3 ] had studied simplicity of $A \times_\alpha \mathbf{Z}$ where $A$ is an abelian $C^*$-algebra and $\mathbf{Z}$ is the integer group. And Jang and Lee [ 8 ] have investigated ideal structure of $A \times_\alpha G$ when $G$ is a discrete group and $A$ is a $C^*$-algebra. When $G$ is a compact group, Gootman, Lazar, and Peligrad showed that $A \times_\alpha G$ is simple if and only if $A$ is $G$-simple and $\widetilde{\Gamma}(\alpha) = \widehat{G}$ by the using the extension of the

Received December 7,1994
Partially suportted by University of Ulsan Research Fund

Connes' spectrum. The problems of the ideal structure of $C^*$-crossed products are very important and interesting, but have not yet been completely solved.

The stable rank of $C^*$-algebras looks like the theory of dimension of $C^*$-algebras. Since $C^*$-algebras are profitably thought of as non-commutative locally compact spaces, with the finitely generated projective module being the appropriate generalization of vector bundle, it would be natural to look for stability result for $C^*$-algebras. But there has been litte discussion of stability properties, presumably in part for lack of an appropriate concept of dimension for $C^*$-algebra. In this sense, the theory of stable rank is to introduce a concept of dimension for $C^*$-algebra which directly generalizes the classical concept of dimension for compact spaces.

## 2. Preliminaries

Let $(A,G,\alpha)$ be a $C^*$-dynamical system and $K(G,A)$ be the norm $*$-algebra of all $A$-valued continuous functions with compact support endowed with the following involution, norm, and twisted convolution as products:

$$x^*(g) = \Delta(g^{-1})\alpha_g(x(g^{-1})^*),$$

$$\|x\|_1 = \int_G \|x(g)\| dg,$$

$$xy(t) = \int_G x(g)\alpha_g(y(g^{-1}t))dg,$$

where $\Delta$ is the modular function. We call the $C^*$-envelope of $L^1(G,A)$ the $C^*$-crossed product of $A$ and $G$ with respect to the action $\alpha$ and write it as $A \times_\alpha G$. Let $(\pi, H)$ be a representation of $A$ and define a covariant representation $(\pi_\alpha, \lambda, L^2(G,H))$

$$(\pi_\alpha(x)\xi)(t) = \pi(\alpha_{t^{-1}}(x))\xi(t)$$

$$(\lambda_g)\xi(t) = \xi(g^{-1}t)$$

for every $x \in A$, $t, g \in G$ and $\xi \in L^2(G,H)$. The regular representation $(\pi_\alpha \times \lambda, L^2(G,H))$ of $A \times_\alpha G$ induced by $(\pi, H)$ is defined such as

$$((\pi_\alpha \times \lambda)y)\xi)(t) = \int_G \pi_\alpha(y(g))\lambda_g\xi(t)dg$$

for every $y \in K(G, A)$ and $\xi \in L^2(G, H)$. Let $(\rho, H)$ denote the universal representation of $A$. The reduced crossed product of $A$ and $G$ is the $C^*$-algebra $(\rho_\alpha \times \lambda)(A \times_\alpha G)$ denoted by $A \times_{\alpha r} G$. If $G$ is amenable, $A \times_\alpha G$ is equal to $A \times_{\alpha r} G$.

## 3. Simplicity of $C^*$-crossed products

Let $A$ be a $C^*$-algebra and $S$ be the state space of $A$. For each state $\phi$ in $S$, let $(\pi_\phi, H_\phi, \xi_\phi)$ denote the cyclic representation associated with $\phi$. For a subset $F$ of $S$ form the Hilbert space $H_F = \oplus_{\phi \in F} H_\phi$ and the representation $\pi_F = \oplus_{\phi \in F} \pi_\phi$ on $H_F$. We say that the space $H_S = \oplus_{\phi \in S} H_\phi$ is the universal Hilbet space and $\pi_S = \oplus_{\phi \in S} \pi_\phi$ is the universal representation. The enveloping von Neumann algebra of $A$ is the strong closure of $\pi_S(A)$. It will hence forth be denoted by $A''$. For each subset $M$ of $B(H)$ let $M'$ denote the commutant of $M$, i.e.

$$M' = \{x \in B(H) \mid xy = yx \qquad \text{for } y \in M\}$$

Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a discrete group. $(A \times_\alpha G)''$ denotes the enveloping von Neumann algebra of $C^*$-crossed product $A \times_\alpha G$. If $M$ is a *-subalgebra of $B(H)$, $Z(M)$ denotes its center, i.e.

$$Z(M) = M \cap M'$$

Let $(A, G, \alpha)$ be a $C^*$-algebra. If $G$ is discrete, the bitransposed action $\alpha''$ induces the $W^*$-dynamical system $(A'', G, \alpha'')$. It is said that $G$ or the action $\alpha''$ acts centrally freely on $A''$ if for any $a \in A''$ and $g \neq e$, where $e$ is the identity of $G$, the condition $ca = a\alpha''_g(c)$ for every central element $c \in A''$ implies that $a = 0$. The free actions are related to the relative commutant property. We consider the similar property for the enveloping von Neumann algebra $(A \times_\alpha G)''$ of $A \times_\alpha G$ as follows ;

$$Z((A \times_\alpha G)'') \subset A'' \ldots (1).$$

LEMMA 3.1. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a discrete group. If the property (1) is satisfied, then $(A \times_\alpha G)''$ is *-isomorphic to the $W^*$-crossed product $A'' \times_{\alpha''} G$.*

*Proof.* Let $\pi : A \times_\alpha G \to B(H)$ be the universal representation of $A \times_\alpha G$. There exists a covariant representation $(\rho, \mu, H)$ such that

$$\pi(f) = (\rho \times \mu)(f) = \sum_{s \in G} \rho(f(s))\mu_s$$

where $\rho : A \to B(H)$ is a representation, $\mu : G \to B(H)$ is a unitary representation, and $f \in l^1(G, A)$. Since $G$ is discrete, we can identify $A$ as a subalgebra of $A \times_\alpha G$, then $\rho = \pi|_A$ is faithful. If $\widetilde{\rho} : A'' \to B(H)$ is the normal extention of $\rho$, then $\widetilde{\rho}(A'')$ is the strong closure of $\rho(A)$ which is isomorphic to $A''$. So $A''$ can be identified $\widetilde{\rho}(A'')$ as subalgebra of $(A \times_\alpha G)''$. Let $\mu : A \to B(H_U)$ be the universal representation of $A$ and $\lambda$ be the regular representation of $G$ on $B(l^2(G, H))$. Define $\mu_\alpha \times \lambda : A \times_\alpha G \to B(l^2(G, H))$ by

$$(\mu_\alpha \times \lambda)(f) = \sum \mu_\alpha(f(s))\lambda_s$$

for $f \in l^1(G, A))$. Let $\tau$ be the $\sigma$-weakly continuous extention of $\mu_\alpha \times \lambda$ to $(A \times_\alpha G)''$. Since the $\sigma$-weak closure of $(\mu_\alpha \times \lambda)(A \times_\alpha G) = A'' \times_{\alpha r} G$, we have only to show that $\tau$ is injective. Put $Ker\tau = I$. Since $I$ is $\sigma$-weakly closed ideal of $(A \times_\alpha G)''$, there exists a central projection $p$ of $(A \times_\alpha G)''$ such that

$$I = (A \times_\alpha G)'' p.$$

By the property (1), there exists an element $q \in A''$ such that

$$p = \tilde{\rho}(q).$$

Since $\tau(\tilde{\rho}(x))\xi)(s) = \alpha''_{s^{-1}}(x)\xi(s)$ for $\xi \in L^2(G, H)$ and $s \in G$,

$$\tau(\rho(q)) = \tau(p) = 0$$

implies thar $q = 0$ Hence $p = 0$ and $\tau$ is injective.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. It is said that $A$ is $G$-simple if $A$ has no non-trivial $\alpha$-invariant closed two sided ideal of $A$.

THEOREM 3.2. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system. If the $C^*$-crossed product $A \times_\alpha G$ is simple, then $A$ is $G$-simple.*

*Proof.* Assume that $A$ is not $G$-simple. Let $I$ be an $\alpha$-invariant norm closed two sided ideal of $A$. Since $I$ is $\alpha$-invariant, we can consider the $C^*$-dynamical system $(I, G, \alpha|_I)$ and $C^*$-crossed product $I \times_\alpha G$. By Lemma 2 of [ 8 ], $I \times_\alpha G$ is a norm closed ideal of $A \times_\alpha G$.

The above theorem says that $G$-simplicity is the necessary condition of simplicity of $C^*$-crossed products.

THEOREM 3.3. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system and let $G$ be a discrete group. Assume that $A$ is $G$-simple and the property (1) is satisfied. Then the $C^*$-crossed product $A \times_\alpha G$ is simple.*

*Proof.* Let $J$ be a non-zero norm closed two sided ideal of $A \times_\alpha G$. The $\sigma$- weak closure $\bar{J}^{\sigma w}$ of $J$ in $(A \times_\alpha G)''$ is the $\sigma$-weak closed two sided ideal of $(A \times_\alpha G)''$. So there exists a projection $e_0$ in the center of $(A \times_\alpha G)''$ such that

$$\bar{J}^{\sigma w} = (A \times_\alpha G)'' e_0.$$

By the property (1), $e_0$ is contained in $A''$. Let $P$ be the conditional expectation from $A'' \times_{\alpha''} G$ onto $A''$. Then $P$is a faithful normal positive linear map. By the Lemma 3.1 there exists a isomorphism

$$\phi : (A \times_\alpha G)'' \to A'' \times_{\alpha''} G.$$

Let $\{e_i\}_{i \in I}$ be an approximate unit of $J$. Then $e_0$ is the least upper bound of $\{e_i\}$. Since $e_i$ exists in $J$ for every $i \in I$, $P(e_i)$ is contained in $A$ for every $i \in I$. Since $e_i \leq e_0$ for every $i \in I$, we get for every $i \in I$

$$P(e_i)e_0 = P(e_i).$$

Therefore $P(e_i)$ is contained in $J \cap A$ for every $i \in I$. $J \cap A$ is non-zero $\alpha$-invariant ideal of $A$. Since $A$ is $G$-simple, we have

$$J \cap A = A.$$

So $J = A \times_\alpha G$.

THEOREM 3.4. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a discrete group If the property (1) is satisfied, $A$ is $G$-prime if and only if the $C^*$-crossed product $A \times_\alpha G$ is prime.*

*Proof.* Suppose that $A$ is $G$-prime. Let $J_1$ and $J_2$ be non-zero closed two sided ideals of $A \times_\alpha G$. The $J_1 \cap A$ and $J_2 \cap A$ are closed two-sided ideals of $A$. Since $A$ is $G$-prime,

$$(J_1 \cap A) \cap (J_2 \cap A)) \neq \{0\}.$$

Thus we have $J_1 \cap J_2 \neq \{0\}$. Conversely suppose that $A \times_\alpha G$ is prime. Let $I_1$ and $I_2$ be non-zero $\alpha$-invariant closed two-sided ideals of $A$. Then $I_1 \times_\alpha G$ and $I_2 \times_\alpha G$ are closed two sided ideals of $A \times_\alpha G$. Since $A \times_\alpha G$ is prime,

$$(I_1 \times_\alpha G) \cap (I_2 \times_\alpha G) \neq \{0\}.$$

As in the proof of Theorem 3.3

$$(I_1 \times_\alpha G) \cap (I_2 \times_\alpha G)) \cap A \neq \{0\}.$$

Since $(I_j \times_\alpha G) \cap A = I_j$, $A$ is $G$-prime.

## 4. Stable rank of $C^*$-crossed products

M.A. Rieffel [ 9 ] introduced and studied the notion of topological stable rank of $C^*$-algebras. He made the notion of dimension for $C^*$-algebras by noticing the following standard theorem from classical dimension theory for compact spaces [ 1 ]. Let $X$ be a compact space. Then the dimension of $X$ is the least integer $n$ such that every continuous function from $X$ into $R^{n+1}$ can be approximately arbitrary closed by functions which do not contain the origin in their range. Now a map $f$ from $X$ to $R^{n+1}$ is just an $(n+1)$ tuple $f_1, \ldots, f_{n+1}$ of real valued functions, and the condition that $f$ miss the origin is the condition that all the $f_i$ nowhere takes the values 0 simultaneously.

If we let $C_R(X)$ denote the Banach algebra of real valued functions on $X$, this last condition is equivalent by the Stone Weierstrass theorem to the condition that the ideal in $C_R(X)$ generated by all the $f_i$ is $C_R(X)$ itself. For any ring with identity, we let $Gen^n(A)$ denote the set of n-tuple of elements of $A$ which generates $A$ as a two sided ideal. Let $A$ be a Banach algebra with identity. By the topological stable rank of $A$, denoted $sr(A)$, we mean the least integer $n$ such that $Gen^n(A)$ is dense in $A^n$ for the product topology. If no such integer exist, we let $sr(A) = \infty$. If $A$ does not have an identity element, then its topological stable ranks are defined to be those for the Banach algebra $\tilde{A}$ obtained from $A$ by adjoining an identity element.

One of the most important problm for topological stable ranks is to construct simple $C^*$-algebras whose topological stable rank is to be given a positive integer. In this section, we consider the topological stable rankof $C^*$-crossed products.

LEMMA 4.1 ( [ 10 ]). *Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a compact abelian. Then we have that*

$$min\,(sr(A^\alpha), 2) \leq sr(A \times_\alpha G) \leq sr(A^\alpha).$$

THEOREM4.2. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a compact abelian group. If $\alpha$ is topologcally transitive, $sr(A \times_\alpha G) \leq 1$.*

*Proof.* Suppose that there exists a non-scalar positive element $x$ in $A^\alpha$. We can choose continuous real valued functions $f(\lambda)$ and $g(\lambda)$ on $spec(x)$ such that

$$supp(f) \cap supp(g) = \emptyset$$

where $supp(f)$ means the support of f. Then by the functional calculus, $f(x)$ and $g(x)$ are contained in $A^\alpha$. Let $B_1$ and $B_2$ be the hereditary $C^*$-subalgebras generated by $f(x)Af(x)$ and $g(x)Ag(x)$ respectively. Since $f(x)$ and $g(x)$ are fixed by $\alpha$, $B_1$ and $B_2$ are $\alpha$-invariant. It is clear that $B_1 B_2 = 0$. So $A^\alpha$ contains only scalar elements. By Lemma 4.1 we have

$$sr(A \times_\alpha G) \leq 1.$$

From [ 7 ] if $\alpha$ is minimal, we can have the same result

Let $G$ be a locally compact abelian group and $H_u$ be the universial Hilbert space of $A$. We consider the unitary representation $U$ on $\widehat{G}$ on $L^2(G, H_u)$ defined such as

$$U_\gamma(\xi)(t) = \gamma(t)\xi(t)$$

for every $\gamma \in \widehat{G}$, $\xi \in L^2(G, H_u)$, and $t \in G$. For each $y$ in $K(G, A)$, we have

$$((U_\gamma y U_\gamma^*)\xi)(t) = ((\gamma y)\xi)(t),$$

where $(\gamma y)(t) = \gamma(t)y(t)$. Let $\widehat{\alpha}_\gamma = Ad_{U_\gamma}$. Then $\widehat{\alpha}_\gamma$ extends to an automorphism on $A \times_\alpha G$ such that $(A \times_\alpha G \times_{\widehat{\alpha}} \widehat{G}, \widehat{G}, \widehat{\alpha})$ becomes a $C^*$-dynamical system, called the dual system of $(A, G, \alpha)$.

THEOREM 4.3. *Let $G$ be a discrete abelian group. Then*

$$min(sr(A), 2) \leq sr(A \times_\alpha G \times_{\widehat{\alpha}} \widehat{G}) \leq sr(A).$$

*Proof.* Since $G$ is discrete, the dual group $\widehat{G}$ is compact. $A$ can be regarded as a subalgebra of $A \times_\alpha G$ because of discreteness of $G$. Furthermore $(A \times_\alpha G \times_{\widehat{\alpha}} \widehat{G})^{\widehat{\alpha}} = A$ by virtue of discreteness of $G$. So by Lemma 4.1 we can have the result.

## References


1. W.Hurewitz and H. Wallman, *Dimension theory*, Princeton University Press, 1941.
2. A.Kishimoto, *Outer automorphisms and reduced crossed products of simple $C^*$-alge -bras*, Comm. Math. Phys. 81 (1981), 429–435.
3. S. Kwamura and J. Tomiyama, *An equivalence formulatin between topological dynamical systems and corresponding $C^*$-algebras*, preprint.
4. H. Nakamura and Z. Takeda, *On some elementary properties of crossed products of von Neumann algebras*, Proc. Japan Acad. **34** (1958), 489–494.
5. J. von Neumann, *On rings of operators, III*, Ann. of Math. **1941**, 94–161.
6. D. Olesen and G. K. Pedersen, *Applications of the Connes' spectrum to $C^*$ dynamical systems I*, J. Funct. Anal. **36** (1978), 179–197.
7. Sun Young Jang and Sa Ge Lee, *Topological Transitivity of compact actions on $C^*$-alge -bras*, Proc. Amer. Math. Soc. **111** (1990), 741–749.
8. Sun Young Jang and Sa Ge Lee, *Simplicity of $C^*$-crossed products of $C^*$-algebras*, Proc. Amer. Math. Soc. **118** (1994), 863–867.
9. M.A. Rieffel, *Dimension and stable rank in the K-theory of $C^*$-algebras*, Proc. London Math. Soc (1983), 301–333
10. H. Takai, *Stable rank of Crossed products by compact abelian groups*, preprint.



Department of Mathematics
University of Ulsan
Ulsan, 680–749, KOREA