PURE SIMPLICITY OF A GROUP  
OVER ITS ENDOMORPHISM RING  

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1. Introduction  
Given any associative ring $R$, let $M$ be a right $R$-module with a submodule $N$ and let $K$ be a left $R$-module. Then we may form the following sequence (not necessarily exact)  

$$0 \rightarrow N \otimes_R K \xrightarrow{i \otimes 1_K} M \otimes_R K$$  
where $i$ is the inclusion homomorphism from $N$ into $M$ and $1_K$ is the identity homomorphism on $K$. If, for a given $K$, the sequence is exact for all $N$ and $M$, then $K$ is said to be left flat: if $M$ and $N$ are given and the sequence is exact for all $K$, then $N$ is called a pure submodule of $M$. If an $R$-module has no nontrivial pure submodules then it is called pure simple. It is well known that an abelian group $G$ forms a module over its endomorphism ring $E(G)$ where $E(G)$ is the set of all endomorphism of $G$. In this short paper, we will find a necessary and sufficient condition under that an abelian torsion group $G$ is pure simple as $E(G)$-module.  

2. Results  
In order to describe this condition we require the following proposition which was proved by P.M.Cohn.  

**Proposition.** Let $M$ be a right $R$-module. Then a submodule $N$ of $M$ is a pure submodule of $M$ if and only if, for any finite sets of elements $m_i \in M, n_j \in N$ and $r_{ij} \in R (i = 1, \cdots m; j = 1, 2, \cdots, n)$, the relations $n_j = \sum m_i r_{ij}$ imply the existence of elements $a_i \in N$ such that $n_j = \sum a_i r_{ij}$.  

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Proof. See theorem 2.4. of [1].

From the proposition, we can have the following examples.

Examples.

(1) Let $p$ be prime. Clearly $\mathbb{Z}_p^n$ is a pure simple as $E(\mathbb{Z}_p^n)$-module. In fact every subgroup of $\mathbb{Z}_p^n$ is a cyclic subgroup generated by $p^k$. Let $\phi_{p^k}$ be an endomorphism of $\mathbb{Z}_p^n$ defined by $(\bar{x})\phi_{p^k} = xp^k$. Then $\bar{1}\phi_{p^k} = p^k$, but $\bar{x}\phi_{p^k} \neq p^k$ for every $\bar{x} \in \langle p^k \rangle$. Thus by proposition $\mathbb{Z}_p^n$ is pure simple.

(2) Let $p$ and $q$ be distinct primes. Then a direct sum of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ is not pure simple because $\mathbb{Z}_p \oplus 0$ and $0 \oplus \mathbb{Z}_q$ are pure submodules of $\mathbb{Z}_p \oplus \mathbb{Z}_q$. We know that $E(\mathbb{Z}_p \oplus \mathbb{Z}_q) = \begin{pmatrix} E(\mathbb{Z}_p) & 0 \\ 0 & E(\mathbb{Z}_q) \end{pmatrix}$ If for every $j$, $\sum_i (\alpha_{ij}, \beta_{ij}) \begin{pmatrix} \alpha_{ij} & 0 \\ 0 & \beta_{ij} \end{pmatrix} = (\bar{z}_j, \bar{0}) \in \mathbb{Z}_p \oplus 0$, then for every $j$, $\sum_i \bar{b}_i \beta_{ij} = \bar{0}$ and $\sum_i \bar{a}_i \alpha_{ij} = \bar{z}_j$. Thus for every $j$, $\sum_i (\bar{a}_i, \bar{0}) \begin{pmatrix} \alpha_{ij} & 0 \\ 0 & \beta_{ij} \end{pmatrix} = (\bar{z}_j, \bar{0})$. Hence $\mathbb{Z}_p \oplus 0$ is a pure submodule by proposition.

Using these examples, we can get the following theorem.

Theorem. Let $G$ be an abelian torsion group. Then $G$ is pure simple as $E(G)$-module if and only if $G$ is a $p$-group.

Proof. Let $G$ be pure simple as $E(G)$-module. Since every abelian torsion group is isomorphic to $\oplus Gp$ where $Gp$ is a $p$-group [2]. From above example (2) we know that $G$ is a $p$-group because $Gp \oplus Gq$ is not pure simple if $p \neq q$. Conversely, we assume that $G$ is a $p$-group. Then $G = A \oplus D$ where $A$ is the reduced subgroup and $D$ is the divisible subgroup of $G$ respectively. We consider the following four cases.

Case 1. If $A = 0$, then $G$ is a divisible $p$-group and isomorphic to a direct sum of copies of $Z(p^\infty)$. Let $\phi_p \in End(G)$ defined by $x \phi_p = xp$ and let $B$ be a nontrivial subgroup of $G$. Then $\pi_\alpha B \neq Z(p^\infty)$ for some index $\alpha$ where $\pi_\alpha$ is a projection. In this case $\pi_\alpha B$ is generated by $\frac{1}{p^k}$ for some $k$. there exists $x \in G$ such that $\pi_\alpha(x \phi_p) = \frac{1}{p^k}$ that is
$x \phi_p \in B$. But there are no elements $y$ in $B$ such that $\pi_\alpha(y \phi_p) = \frac{1}{p^x}$. So $G$ is simple as $E(G)$-module.

Case 2. If $D = 0$, then $G$ is isomorphic to $Z_{p^{i_1}} + Z_{p^{i_2}} + \cdots + Z_{p^{i_n}}$ ($i_1 \leq i_2 \leq \cdots \leq i_n$). Then example (1) shows that $G$ is pure simple as $E(G)$-module.

Case 3. If $A \neq 0$, $D \neq 0$ and rank of $D$ is 1, then $A$ is isomorphic to $Z_{p^{i_1}} + Z_{p^{i_2}} + \cdots + Z_{p^{i_n}}$ ($i_1 \leq i_2 \leq \cdots \leq i_n$) and $D$ is isomorphic to $Z(p^\infty)$. In this case we know that $E(G) = \begin{pmatrix} E(A) & Hom(A, D) \\ 0 & E(D) \end{pmatrix}$ where $(a, d) \begin{pmatrix} \alpha \\ 0 \end{pmatrix}^{\beta}$, $\alpha \in (A), \beta \in Hom(A, D), \gamma \in E(D)$. Note that $Hom(D, A) = 0$. Let $\beta \in Hom(A, D)$ define by the following

\[
\begin{cases} 
(0, \overline{0}, \cdots, \overline{1}) \beta = \frac{1}{p^{i_n}} & \text{and} \\
(a) \beta = 0 & \text{if } a \in Z_{p^{i_1}} \oplus Z_{p^{i_2}} \oplus \cdots \oplus Z_{p^{i_{n-1}}} \oplus 0
\end{cases}
\]

and let $\phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. Then we can get the following equation

\[
((0, \overline{0}, \ldots, \overline{1}), 0) \phi = (0, \frac{1}{p^{i_n}}) \in D
\]

But we know that there are no elements of $D$ such that $(0, d) \phi = (0, \frac{1}{p^{i_n}})$. In fact we know that for every $d \in D$, $(0, d) \phi = (0, 0)$. Thus $0 \oplus D$ is not pure submodule of $G$ as $E(G)$-module. And $A \oplus 0$ is not characteristic subgroup of $G$. Hence we know that there are no pure submodules of $G$ as $E(G)$-module.

Case 4. If rank of $D$ is larger than 1, similarly we can know that $0 \oplus D$ is not pure submodule of $G$ by Case 3.

From the above theorem, naturally we can get the following corollary.

**Corollary.** Every abelian torsion group $G$ is pure semisimple as $E(G)$-module.

**Proof.** Since $G = \oplus G_p$ and each $G_p$ is pure simple as $E(G)$-module we know that $G$ is pure semisimple as $E(G)$-module.
References


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