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ON THE SPECTRAL GEOMETRY OF CLOSED MINIMAL TOTALLY REAL SUBMANIFOLDS IN A COMPLEX SPACE FORM

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1. Introduction

The spectral geometry for the second order operators arising in Riemannian geometry has been studied by many authors [1,3,4,5,7,10,11]. Among them, the spectral geometry of the normal Jacobi operator for minimal submanifolds was studied in [1,4,5,6]. The normal Jacobi operator arises in the second variation formula for the functional area. This formula can be expressed in terms of an elliptic differential operator $\mathcal{J}$ (called the *normal Jacobi operator*) defined on the cross section $\Gamma(NM)$ of the normal bundle of the isometric minimal immersion $f : M \longrightarrow N$, which is defined by $\mathcal{J} = \tilde{\Delta} + \tilde{R} - S$, where $\tilde{\Delta}$ is the rough Laplacian on NM and $\tilde{R}$ and S are linear transformations of NM defined by means of a partial *Ricci operator* $\tilde{R}$ of N and of the second fundamental form and its transpose, respectively.

The purpose of the present paper is to study the spectral geometry for totally real submanifolds in a manifold of constant holomorphic sectional curvature.

The spectral geometry for the Jacobi operator of the energy of a harmonic map was studied in [8,10,11].

2.Preliminaries

For a Riemannian manifold M which is isometrically immersed in a Riemannian manifold N with the Riemannian metric g, the formulas of Gauss and Weingarten are respectively given by

$$(2.1) \qquad {}^N\nabla_X Y = \nabla_X Y + B(X,Y), \qquad {}^N\nabla_X V = -A^V X + D_X V$$

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for vector fields X, Y tangent to M and a normal vector field V, where ∇ be the Levi-Civita connection on M, and A and B are called the *second fundamental forms* of M, which are related by $g(V, B(X,Y)) = g(A^V(X), Y)$.

Furthermore, we can consider A as a cross section of the Riemannian vector bundle $Hom(NM, SM)$, where SM is the bundle of symmetric transformations of the tangent bundle TM and NM is the normal bundle of M in N. Then $S := {}^tA \circ A \in \Gamma(Hom(NM, NM))$, where $\Gamma(\bullet)$ denotes the space of smooth sections of $\bullet$. Henceforth we adopt the following notations ;

σ := the trace of ${}^tA \circ A$(*i.e.*, the square norm of A),

l_n := the trace of $S \circ S$(*i.e.*, the square norm of S),

k_n := the square norm of the curvature tensor of the normal connection,

t := the square norm of the covariant derivative of the second fundamental form A.

A submanifold M of an almost complex manifold (N, J) is said to be *totally real* provided that the almost complex structure J of N maps tangent vectors to M into normal vectors. A Kaehler manifold is a complex space form of constant holomorphic sectional curvature k, denoted by $N(k)$, if its curvature tensor R satisfies

$$R(X,Y)Z = \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\},$$

where X, Y, Z are vector fields in N.

We denote by $CP^n(k)$ the complex projective space of real dimension $2n$ with constant holomorphic sectional curvature k, and $RP^n(\frac{k}{4})$ the real projective space with constant sectional curvature $\frac{k}{4}$. Then there is a natural embedding of real projective space $RP^n(\frac{k}{4})$ as totally real, totally geodesic submanifold of $CP^n(k)$.

Now we introduce the Weyl conformal curvature tensor C and the Einstein tensor G on M, which are respectively defined by

$$C(X,Y)Z = R(X,Y)Z + \frac{1}{n-2}\{\rho(X,Z)Y - \rho(Y,Z)X + g(X,Z)QY - g(Y,Z)QX\} - \frac{1}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\},$$

$$G(X,Y) = \rho(X,Y) - \frac{1}{n}g(X,Y)\tau$$

for any vector fields X, Y, Z on M, where $g(QX,Y) = \rho(X,Y)$, and for a local orthonormal frame field $\{e_1, \cdots, e_n\}$ $\rho(X,Y) := \sum_{i=1}^n g(R(e_i,X)Y, e_i)$ and τ are the Ricci tensor and the scalar curvature on M, respectively. Then we have

$$|C|^2 = |R|^2 - \frac{4}{n-2}|\rho|^2 + \frac{2}{(n-1)(n-2)}\tau^2, \tag{2.2}$$

$$|G|^2 = |\rho|^2 - \frac{1}{n}\tau^2. \tag{2.3}$$

$G = 0$ holds if and only if M is Einstein. $C = 0$ and $G = 0$ hold if and only if M has a constant sectional curvature ($n \geq 4$).

Let $\tilde{\mathcal{R}}$ be the partial Ricci transformation, which is defined by

$$\tilde{\mathcal{R}}(V) := \sum_{i=1}^{n} \{R(e_i, V)e_i\}^{\perp},$$

where V is a normal vector field and $\perp$ denotes the normal part of a vector with respect to the metric g.

Now we consider the differential operator $\mathcal{J}$, which is usually called the *normal Jacobi operator*, defined by

$$\mathcal{J} = \tilde{\Delta} + \tilde{R} - S,$$

where $\tilde{\Delta} = -\sum_{i=1}^n (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i})$

Throughout this paper M will denote a closed (compact without boundary) manifold. In fact the operator $\mathcal{J}$ arising from the second variation formula of M is self-adjoint, elliptic of second order, and has a discrete spectrum as consequence of compactness of M.

From now on we assume that M denotes an n-dimensional totally real submanifold of a $2n$-dimensional Kaehler manifold. Then we obtain from (2.1)

$$D_X(JY) = J\nabla_X Y, \tag{2.4}$$

$$JB(X,Y) = -A^{JY}X, \tag{2.5}$$

$$g(B(X,Y),JZ) = g(B(X,Z),JY) \tag{2.6}$$

for any vector fields X, Y, Z tangent to M.

If an n-dimensional totally real submanifold M of a $2n$-dimensional complex space form $N(k)$ is minimal, then the Simon's type formula [cf.12] is given by

$$\frac{1}{2}\tilde{\Delta}\sigma = t - \tilde{k_n} - l_n + \frac{k}{4}(n+1)\sigma, \tag{2.7}$$

where $\tilde{k_n} := -\sum_{a,b} Tr([A^a, A^b]^2), A^a := A^{e_a}$, $\{e_a : a = n+1, \cdots, 2n\}$ a local orthonormal basis of the normal space N_xM at $x \in M$, $[A^a, A^b] = A^a \circ A^b - A^b \circ A^a$.

3. The calculation of spectral invariants

In this section we apply the normal Jacobi operator $\mathcal{J}$ acting on $\Gamma(NM)$ to the Gilkey's results.

Now consider the semigroup $e^{-t\mathcal{J}}$ given by

$$e^{-t\mathcal{J}}V(x) = \int_M K(t,x,y,\mathcal{J})V(y)dv_g(y),$$

where $K(t,x,y,\mathcal{J}) \in Hom(N_yM, N_xM)$ is the kernel function and dv_g denotes the volume element of M with respect to g. Then we have asymptotic expansions for L^2-trace

$$Tr(e^{-t\mathcal{J}}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-\frac{2n}{2}} \sum_{j=0}^{\infty} t^j a_j(\mathcal{J}) \quad (t \downarrow 0^+), \tag{3.1}$$

where $2n$ denotes the real dimension N, and each $a_j(\mathcal{J})$ is the spectral invariants of $\mathcal{J}$, which depends only on the discrete spectrum ;

$$Spec(M, g) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}.$$

Applying the normal Jacobi operator $\mathcal{J}$ to the Gilkey's results [4,p.327] , we obtain

THEOREM [cf. 3,4].

(i) $$a_0(\mathcal{J}) = q \cdot Vol(M, g),$$

(ii) $$a_1(\mathcal{J}) = \frac{q}{6} \int_M \tau dv_g + \int_M Tr(E) dv_g,$$

(iii) $$a_2(\mathcal{J}) = \frac{q}{360} \int_M (5\tau^2 - 2|\rho|^2 + 2|R|^2) dv_g + \frac{1}{360} \int_M \{-30k_n + Tr(60\tau E + 180E^2)\} dv_g,$$

where q is the codimension n and $E := -\tilde{R} + S$.

If M is an n-dimensional, minimal, totally real submanifold of a complex space form $N(k)$ with dimension $2n$, then we obtain

$$\tau = \frac{k}{4} n(n-1) - \sigma, \tag{3.2}$$

$$Tr(E) = \frac{k}{2} n(n+1) - \tau, \tag{3.3}$$

$$Tr(E^2) = \frac{k^2}{16} n(n+3)^2 + \frac{k}{2}(n+3)\sigma + l_n, \tag{3.4}$$

$$k_n = \tilde{k}_n - \frac{k^2}{8} n(n-1) + k\tau, \tag{3.5}$$

where (3.2) follows from the equation of Gauss, (3.3) and (3.4) from the definition of E, and (3.5) from the equation of Ricci, (3.2) and (2.6)

Substituting (3.2) $\sim$ (3.5) into THEOREM, we get

THEOREM 1. *Let M be an n-dimensional compact, minimal, totally real submanifold of a $2n$-dimensional complex space form $N(k)$ with constant holomorphic sectional curvature k. Then the coefficients $a_0(\mathcal{J}), a_1(\mathcal{J})$ and $a_2(\mathcal{J})$ of the asymptotic expansion for the normal Jacobi operator $\mathcal{J}$ are respectively given by*

$$a_0(\mathcal{J}) = nVol(M,g), \tag{3.6}$$

$$a_1(\mathcal{J}) = \frac{n-6}{6}\int_M \tau dv_g + \frac{k}{2}n(n+1)Vol(M,g), \tag{3.7}$$

$$\begin{aligned} a_2(\mathcal{J}) = \frac{1}{360}\int_M &[2n|R|^2 - 2n|\rho|^2 \\ &+ 5(n-12)\tau^2 - 30\tilde{k}_n + 180l_n]dv_g \\ &+ \frac{k}{12}\int_M (n^2-2n-10)\tau dv_g + a_0 Vol(M,g), \end{aligned} \tag{3.8}$$

where a_0 is a number determined by n and k.

COROLLARY 1. *Under the same situations as stated in Theorem 1, the following quantities are its spectral invariants when n is not equal to 6.*

(1) $dim M,\ Vol(M,g),\ \int_M \tau dv_g,\ \int_M (\tilde{k}_n + l_n - t)dv_g,$

(2) $\int_M \sigma dv_g,$

(3) $\frac{n}{180}\int_M (|R|^2 - |\rho|^2)dv_g + \frac{n-12}{72}\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6l_n - \tilde{k}_n)dv_g,$

(4) $\frac{n}{180}\int_M (|C|^2 + \frac{6-n}{n-2}|G|^2)dv_g + a_1\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6l_n - \tilde{k}_n)dv_g,$

(5) $\frac{n}{180}\int_M (|C|^2 + \frac{6-n}{n-2}|G|^2)dv_g + a_1\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6t - 7\tilde{k}_n)dv_g,$

where $a_1 = \dfrac{5n^2 - 67n + 66}{360(n-1)}$.

Proof. (1) and (2) follow from (2.7), (3.1), (3.6) and (3.7). Substituting (2.2) and (2.3) into (3.8), we obtain (4). (5) follows from (4) and the fourth spectral invariant of (1). *Q.E.D.*

4. Some Results

From now on, we consider n-dimensional, compact, miniaml, totally real submanifolds M and M' of $N(k)$ with dimension $2n$.

First of all we have from (2) of Corollary 1

PROPOSITION 1. *Assume that $Spec(M,\mathcal{J}) = Spec(M',\mathcal{J}')$. Then if M is totally geodesic, so does M'.*

PROPOSITION 2. *Assume that $Spec(M,\mathcal{J}) = Spec(M',\mathcal{J}')$ and $\int_M (l_n - t)dv_g \le \int_{M'}(l'_n - t')dv_{g'}$. Then the second fundamental forms on M commute each other if and only if the second fundamental forms on M' commute each other and $\int_M (l_n - t)dv_g = \int_{M'}(l'_n - t')dv_{g'}$.*

Proof. This follows from (1) of Corollary 1. *Q.E.D.*

We get from (5) of Corollary 1

PROPOSITION 3. *Let M and M' be Einstein. Assume that $Spec(M,\mathcal{J}) = Spec(M',\mathcal{J}')$, $\int_M (6t - 7\tilde{k}_n)dv_g \le \int_{M'}(6t' - 7\tilde{k}'_n)dv_{g'}$. Then M has a constant curvature $\tilde{k}$ if and only if M' has the same constant curvature $\tilde{k}$ and $\int_M (6t - 7\tilde{k}_n)dv_g = \int_{M'}(6t' - 7\tilde{k}'_n)dv_{g'}$.*

PROPOSITION 4. *Assume that $Spec(M,\mathcal{J}) = Spec(M',\mathcal{J}')$. If M has a constant curvature $\tilde{k}$, and M' is Einstein and if $\int_M (6l_n - \tilde{k}_n)dv_g \le \int_{M'}(6l'_n - \tilde{k}'_n)dv_{g'}$, then M' has the same constant curvature $\tilde{k}$ and $\int_M (6l_n - \tilde{k}_n)dv_g = \int_{M'}(6l'_n - \tilde{k}'_n)dv_{g'}$.*

Proof. It follows from (4) of Corollary 1. *Q.E.D.*

PROPOSITION 5. *Let M be an n-dimensional compact minimal, totally real submanifold of $CP^n(k)$ Assume that $Spec(M,\mathcal{J}) = Spec(RP^n(\frac{k}{4}),\mathcal{J}')$. Then M is a totally geodesic $RP^n(\frac{k}{4})$.*

Proof. Proposition 1 implies that M is totally geodesic in $CP^n(k)$. Then M is $RP^n(\frac{k}{4})$(cf.[9]). *Q.E.D.*

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