WEAKLY COMPACT FUNCTIONS

V.P. SINGH, G.I. CHAE AND R. POONIWALA

1. Introduction

Throughout this note, spaces always mean topological spaces unless explicitly stated and we will denote a function $f$ from a space $X$ into $Y$ by $f : X \to Y$, the graph of $f$ by $G_f$, the closure of $U$ by $cl(U)$ and the set $V$ containing $x$ by $V_x$. For definitions and terminologies not explained, we will refer to [1,3,6].

A function $f : X \to Y$ is said to be compact [7] if for each closed and compact set $K \subset Y, f^{-1}(K)$ is closed and compact in $X$; to be $\mathcal{C}$-continuous [3] if for each open set $V \subset Y$ having compact complement, $f^{-1}(V)$ is open in $X$; to have closed graph [5] (resp. strongly closed graph [6]) if for each pair $(x, y) \in X \times Y, y \neq f(x)$, there exist open sets $U_x$ and $V_y$ such that $f(U_x) \cap V_y = \emptyset$ (resp. $f(U_x) \cap cl(V_y) = \emptyset$).

In this note a new concept of a function called weakly compact is defined and investigate relationships between weakly compact functions, graph functions, known functions and the Co-compact space introduced in this note. We will know the fact (Confer to Theorem 2) that most (or more) of results in [3] will be obtained from weakly compactness.

2. W-compact functions and Co-compact spaces

DEFINITION 1. A function $f : X \to Y$ is said to be weakly compact (briefly W-compact) if for each closed and compact set $K$ of $Y, f^{-1}(K)$ is closed in $X$.

Every continuous function is W-compact but its converse need not be true. For example, the identity $I : (R, \mathcal{U}) \to (R, \mathcal{D})$ is W-compact (even compact) but not continuous where $\mathcal{U}$ and $\mathcal{D}$ are respectively the usual and discrete spaces of real numbers. It is interesting to note that weakly compact functions are $\mathcal{C}$-continuous and vice-versa.

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Theorem 2. \( f : X \rightarrow Y \) is W-compact if and only if \( f \) is C-continuous.

Proof. Let \( f \) be C-continuous and \( K \) a closed compact subset of \( Y \). Then \( Y \setminus K \) is open with compact complement. Thus \( f^{-1}(Y \setminus K) = X \setminus f^{-1}(K) \) is open, i.e., \( f^{-1}(K) \) is closed in \( X \) and hence \( f \) is W-compact. Conversely, let \( f \) be W-compact and \( V \) an open set of \( Y \) having compact complement. Then \( f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) \) is closed, i.e., \( f^{-1}(V) \) is open. Thus \( f \) is C-continuous.

Theorem 3. If \( f : X \rightarrow Y \) has the closed graph, then \( f \) is W-compact.

The proof follows directly from [4, Proposition 9, p.200] stating that for any compact subset \( K \subset Y \), \( f^{-1}(K) \) is closed in \( X \) whenever \( f \) has its graph closed. However, the converse of Theorem 3 need not be true as shown by the below example.

Example 4. Let \( X \) be an infinite set with cofinite topology. Then the identity \( I \) on \( X \) is W-compact but its graph is not closed.

Theorem 5. Let \( X \) be regular and \( Y \) compact Hausdorff. If \( f : X \rightarrow Y \) is W-compact, then \( G_f \) is strongly closed and thus closed.

Proof. Let \((x,y) \notin G_f \). Then \( y \neq f(x) \). Since \( Y \) is Hausdorff, there are open sets \( O_y \) and \( W_{f(x)} \) such that \( O_y \cap W_{f(x)} = \emptyset \). Since \( Y \) is compact Hausdorff and \( y \in O_y \), there is an open set \( V_y \) such that \( y \in V_y \subset \text{cl}(V_y) \subset O_y \). Since \( cl(V_y) \) is closed compact (for \( Y \) is compact) and \( f \) is W-compact, \( f^{-1}(cl(V_y)) \) is a closed set of \( X \) not containing \( x \). By regularity of \( X \), there is an open set \( U_x \) such that \( f^{-1}(cl(V_y)) \cap cl(U_x) = \emptyset \). Hence we have \( f(U_x) \cap cl(V_y) = \emptyset \).

Theorem 6. Let \( X \) be regular and \( Y \) be \( T_1 \). If \( f : X \rightarrow Y \) is closed and W-compact, then \( f \) has closed graph.

Proof. For any pair \((x,y) \notin G_f \), i.e., \( x \notin f^{-1}(y) \), (or \( y \neq f(x) \)), \( f^{-1}(y) \) is a closed set not containing \( x \) since \( f \) is W-compact and \( Y \) is \( T_1 \). Since \( X \) is regular, there is an open sets \( U_x \) such that \( f^{-1}(y) \cap cl(U_x) = \emptyset \), i.e., \( f^{-1}(y) \subset X \setminus cl(U_x) \). Since \( f \) is closed, from [2, Theorem 11.2, p.86] there exists an open set \( V_y \) such that \( f^{-1}(y) \subset f^{-1}(V_y) \subset X \setminus cl(U_x) \). So \( V_y \cap f(U_x) = \emptyset \). So \( f \) has a closed graph.

Composition of W-compact functions need not be W-compact. Let \( X, Y \) and \( Z \) be cofinite, discrete and usual spaces of real numbers
respectively. Then \( I_1 \circ I_2 \) is not W-compact for identities \( I_1 : X \to Y \) and \( I_2 : Y \to Z \) even though \( I_1 \) and \( I_2 \) are W-compact. Relationships between W-compact functions and its graph functions are shown.

**Theorem 7.** If \( f : X \to Y \) is continuous and \( g : Y \to Z \) is W-compact, then \( g \circ f \) is W-compact.

The proof is obvious and is thus omitted. It is easy to prove that \( g \circ f \) is W-compact if \( Y \) is a compact space where \( f : X \to Y \) and \( g : Y \to Z \) are W-compact functions.

**Theorem 8.** Let \( f : X \to Y \) be W-compact. Then \( G_f : X \to X \times Y \) where \( G_f(x) = \{(x, f(x)) : x \in X\} \) is W-compact.

**Proof.** By Theorem 2, it suffices to show that for any open set \( U \times V \) in \( X \times Y \) having compact complement, \( G^{-1}(U \times V) \) is open in \( X \). Let \( K = X \times Y \setminus (U \times V) = K = (X \setminus U) \times Y \cup X \times (Y \setminus V) \). Since \( X \times (Y \setminus V) \) is compact for it is a closed subset of the compact set \( K \) and so \( P_Y(X \times (Y \setminus V)) = Y \setminus V \) is compact where \( P_Y \) is the projector on \( Y \). Since \( f \) is W-compact and hence \( f^{-1}(V) \) is open, \( G_f^{-1}(U \times V) = U \cap f^{-1}(V) \) is open in \( X \). Thus \( G_f \) is W-compact.

The converse of Theorem 8 need not be true. In the case that \( X \) is compact we have the following stronger result.

**Theorem 9.** Let \( X \) be a compact space. Then \( f : X \to Y \) is W-compact whenever \( G_f \) is W-compact.

**Proof.** Let \( V \) be an open set of \( Y \) having compact complement. Then it is enough to show that \( f^{-1}(V) \) is open in \( X \). Consider \( X \times Y \setminus (X \times V) = X \times (Y \setminus V) \) is compact, for \( X \) and \( Y \setminus V \) are compact. Since \( G_f \) is W-compact, \( G_f^{-1}(X \times V) = f^{-1}(V) \) is open in \( X \).

**Theorem 10.** Let \( X \) be normal and \( Y \) be \( T_1 \). If \( f : X \to Y \) is surjective and W-compact, then \( Y \) is \( T_2 \).

**Proof.** Let \( x, y \in X, x \neq y \). Then \( \{x\}, \{y\} \) are closed and compact subsets of \( Y \) and thus \( f^{-1}(x), f^{-1}(y) \) are closed in \( X \). By normality of \( X \) there are disjoint open sets \( U_1 \) and \( U_2 \). Since \( f \) is closed, from [2, Theorem 11.2, p. 86] there are \( V_x \) and \( V_y \) such that \( f^{-1}(x) \subset f^{-1}(V_x) \subset U_1 \) and \( f^{-1}(y) \subset f^{-1}(V_y) \subset U_2 \). Since \( U_1 \cap U_2 = \emptyset, f^{-1}(V_x \cap V_y) = \emptyset \). \( V_x \cap V_y = \emptyset \). \( Y \) is a \( T_2 \) space.
**Definition** 11. A space $X$ is said to be Co-compact if for each $x \in X$ and each open set $U_x$ (containing $x$), there exists an open set $V_x$ such that $x \in V_x \subset U_x$ and $X \setminus V_x$ consists of finite number of closed compact subsets of $X$.

**Example** 12. Finite spaces, indiscrete spaces and an infinite space with cofinite topology are some of the examples of Co-compact spaces even though the usual space of real numbers is not Co-compact.

**Theorem** 13. If $f : X \to Y$ is $W$-compact and $Y$ is a Co-compact space, then $f$ is continuous.

*Proof.* Let $x \in X$, $y = f(x)$ and $V$ be any open neighborhood of $y$. Since $Y$ is Co-compact, there is an open set $V_y$ such that $y \in V_y \subset V$ and $Y \setminus V_y$ consists of finite number of closed compact sets, say $C_1, C_2, C_3, \ldots, C_n$. Since each $C_k$ is closed and compact, each $f^{-1}(C_k)$ is closed by $W$-compactness of $f$. Let $C = \bigcup_{k=1}^{n} f^{-1}(C_k)$. Then $C$ is closed in $X$ and $W = X \setminus C$ is an open set containing $x$ such that $W = f^{-1}(Y) \setminus f^{-1}(\bigcup_{k=1}^{n} C_k) = f^{-1}(Y \setminus (Y \setminus V_y)) = f^{-1}(V_y) \subset f^{-1}(V)$. So $f(W) \subset V$. Hence $f$ is continuous.

**References**


Department of Mathematics
Regional College of Ed.
Bhopal 462-013, India

Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea