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WEAKLY COMPACT FUNCTIONS

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1. Introduction

Throughout this note, spaces always mean topological spaces unless explicitly stated and we will denote a function f from a space X into Y by $f : X \to Y$, the graph of f by G_f, the closure of U by $cl(U)$ and the set V containing x by V_x. For definitions and terminologies not explained, we will refer to [1,3,6].

A function $f : X \to Y$ is said to be compact [7] if for each closed and compact set $K \subset Y, f^{-1}(K)$ is closed and compact in X; to be C-continuous [3] if for each open set $V \subset Y$ having compact complement, $f^{-1}(V)$ is open in X; to have closed graph [5] (resp. strongly closed graph [6]) if for each pair $(x, y) \in X \times Y$, $y \neq f(x)$, there exist open sets U_x and V_y such that $f(U_x) \cap V_y = \emptyset$ (resp. $f(U_x) \cap cl(V_y) = \emptyset$).

In this note a new concept of a function called weakly compact is defined and investigate relationships between weakly compact functions, graph functions, known functions and the Co-compact space introduced in this note. We will know the fact (Confer to Theorem 2) that most (or more) of results in [3] will be obtained from weakly compactness.

2. W-compact functions and Co-compact spaces

Definition 1. A fuction $f : X \to Y$ is said to be weakly compact (briefly W-compact) if for each closed and compact set K of Y, $f^{-1}(K)$ is closed in X.

Every continuous function is W-compact but its converse need not be true. For example, the identity $I : (R, \mathcal{U}) \to (R, \mathcal{D})$ is W-compact (even compact) but not continuous where $\mathcal{U}$ and $\mathcal{D}$ are respectively the usual and discrete spaces of real numbers. It is interesting to note that weakly compact functions are C-continuous and vice-versa.

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THEOREM 2. *$f : X \to Y$ is W-compact if and only if f is C-continuous.*

Proof. Let f be C-continuous and K a closed compact subset of Y. Then $Y \setminus K$ is open with compact complement. Thus $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$ is open, i.e., $f^{-1}(K)$ is closed in X and hence f is W-compact. Conversely, let f be W-compact and V an open set of Y having compact complement. Then $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is closed, i.e., $f^{-1}(V)$ is open. Thus f is C-continuous.

THEOREM 3. *If $f : X \to Y$ has the closed graph, then f is W-compact.*

The proof follows directly from [4, Proposition 9, p.200] stating that for any compact subset $K \subset Y$, $f^{-1}(K)$ is closed in X whenever f has its graph closed. However, the converse of Theorem 3 need not be true as shown by the below example.

EXAMPLE 4. Let X be an infinite set with cofinite topology. Then the identity I on X is W-compact but its graph is not closed.

THEOREM 5. *Let X be regular and Y compact Hausdorff. If $f : X \to Y$ is W-compact, then G_f is strongly closed and thus closed.*

Proof. Let $(x, y) \notin G_f$. Then $y \neq f(x)$. Since Y is Hausdorff, there are open sets O_y and $W_{f(x)}$ such that $O_y \cap W_{f(x)} = \emptyset$. Since Y is compact Hausdorff and $y \in O_y$, there is an open set V_y such that $y \in V_y \subset cl(V_y) \subset O_y$. Since $cl(V_y)$ is closed compact (for Y is compact) and f is W-compact, $f^{-1}(cl(V_y))$ is a closed set of X not containing x. By regularity of X, there is an open set U_x such that $f^{-1}(cl(V_y)) \cap cl(U_x) = \emptyset$. Hence we have $f(U_x) \cap cl(V_y) = \emptyset$.

THEOREM 6. *Let X be regular and Y be T_1. If $f : X \to Y$ is closed and W-compact, then f has closed graph.*

Proof. For any pair $(x, y) \notin G_f$, i.e., $x \notin f^{-1}(y)$, (or $y \neq f(x)$), $f^{-1}(y)$ is a closed set not containing x since f is W-compact and Y is T_1. Since X is regular, there is an open sets U_x such that $f^{-1}(y) \cap cl(U_x) = \emptyset$, i.e., $f^{-1}(y) \subset X \setminus cl(U_x)$. Since f is closed, from [2, Theorem 11.2, p.86] there exists an open set V_y such that $f^{-1}(y) \subset f^{-1}(V_y) \subset X \setminus cl(U_x)$. So $V_y \cap f(U_x) = \emptyset$. So f has a closed graph.

Composition of W-compact functions need not be W-compact. Let X, Y and Z be cofinite, discrete and usual spaces of real numbers respectively. Then $I_1 \circ I_2$ is not W-compact for identities $I_1 : X \to Y$ and $I_2 : Y \to Z$ even though I_1 and I_2 are W-compact. Relationships between W-compact functions and its graph functions are shown.

THEOREM 7. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is W-compact, then $g \circ f$ is W-compact.*

The proof is obvious and is thus omitted. It is easy to prove that $g \circ f$ is W-compact if Y is a compact space where $f : X \to Y$ and $g : Y \to Z$ are W-compact functions.

THEOREM 8. *Let $f : X \to Y$ be W-compact. Then $G_f : X \to X \times Y$ where $G_f(x) = \{(x, f(x)) : x \in X\}$ is W-compact.*

Proof. By Theorem 2, it suffices to show that for any open set $U \times V$ in $X \times Y$ having compact complement, $G^{-1}(U \times V)$ is open in X. Let $K = X \times Y \setminus (U \times V) = K = (X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since $X \times (Y \setminus V)$ is compact for it is a closed subset of the compact set K and so $P_Y(X \times (Y \setminus V)) = Y \setminus V$ is compact where P_Y is the projecton on Y. Since f is W-compact and hence $f^{-1}(V)$ is open, $G_f^{-1}(U \times V)$ $= U \cap f^{-1}(V)$ is open in X. Thus G_f is W-compact.

The converse of Theorem 8 need not be true. In the case that X is compact we have the following stronger result.

THEOREM 9. *Let X be a compact space Then $f : X \to Y$ is W-compcat whenever G_f is W-compact.*

Proof. Let V be an open set of Y having compact complement. Then it is enough to show that $f^{-1}(V)$ is open in X. Consider $X \times Y \setminus (X \times V)$ $= X \times (Y \setminus V)$ is compact, for X and $Y \setminus V$ are compact. Since G_f is W-compact, $G_f^{-1}(X \times V) = f^{-1}(V)$ is open in X.

THEOREM 10. *Let X be normal and Y be T_1. If $f : X \to Y$ is surjective and W-compact, then Y is T_2.*

Proof. Let $x, y \in Y, x \neq y$. Then $\{x\}, \{y\}$ are closed and compact subsets of Y and thus $f^{-1}(x), f^{-1}(y)$ are closed in X. By normality of X there are disjoint open sets U_1 and U_2. Since f is closed, from [2, Theorem 11.2, p. 86] there are V_x and V_y such that $f^{-1}(x) \subset f^{-1}(V_x) \subset U_1$ and $f^{-1}(y) \subset f^{-1}(V_y) \subset U_2$. Since $U_1 \cap U_2 = \emptyset, f^{-1}(V_x \cap V_y) = \emptyset$. $V_x \cap V_y = \emptyset$. Y is a T_2 space.

DEFINITION 11. A space X is said to be Co-compact if for each $x \in X$ and each open set U_x (containing x), there exists an open set V_x such that $x \in V_x \subset U_x$ and $X \setminus V_x$ consists of finite number of closed compact subsets of X.

EXAMPLE 12. Finite spaces, indiscrete spaces and an infinite space with cofinite topology are some of the examples of Co-compact spaces even though the usual space of real numbers is not Co-compact.

THEOREM 13. *If $f : X \to Y$ is W-compact anf Y is a Co-compact space, then f is continuous.*

Proof. Let $x \in X$, $y =$ f(x) and V be any open neighborhood of y. Since Y is Co-compact, there is an open set V_y such that $y \in V_y \subset V$ and $Y \setminus V_y$ consists of finite number of closed compact sets, say $C_1, C_2, C_3, ...C_n$. Since each C_k is closed and compact, each $f^{-1}(C_k)$ is closed by W-compactness of f. Let $\mathcal{C} = \cup_{k=1}^{n} f^{-1}(C_k)$. Then $\mathcal{C}$ is closed in X and $W = X \setminus \mathcal{C}$ is an open set containing x such that $W = f^{-1}(Y) \setminus f^{-1}(\cup_{k=1}^{n} C_k) = f^{-1}(Y \setminus (Y \setminus V_y)) = f^{-1}(V_y) \subset f^{-1}(V)$. So $f(W) \subset V$. Hence f is continuous.

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