CONTROLLABILITY PROPERTIES OF DELAY VOLTERRA CONTROL SYSTEM

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1. Introduction

We consider the following delay volterra control system

\[
x_t(\phi : u)(0) = U(t, 0)\phi(0) + \int_0^t U(t, s)\{F(s, x_s(\phi : u), u(s)) + Bu(s)\}\, ds
\]

\[x_0(\theta) = \phi \in C.
\]

Here, let X and U be Hilbert spaces. The state function \(x(t)\), \(0 \leq t \leq T\), takes values in X and the control function \(u\) is given in \(L^2(0, T : U)\) and \(U(t, s)\) is a linear evolution operator on X. Let C be a Banach space of all continuous functions from an interval of the form \(I = [-h, 0]\) to X with the norm defined by supremum. If a function \(u\) is continuous from \(I \cup [0, T]\) to X, then \(u_t\) is an element in C which has point-wise definition \(u_t(\theta) = u(t + \theta)\) for \(\theta \in I\).

We assume that \(F\) is a nonlinear function from \([0, T] \times C \times L^2(0, T : U)\) to X and B is a bounded linear operator from \(L^2(0, T : U)\) to \(L^2(0, T : X)\).

The purpose of this paper is to give some general conclusions on both approximate controllability and exact reachability.

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2. Preliminaries and Estimation

The norm of the space $L^2(0, T : X)$ or $L^2(0, T : U)$ is denoted by $\| : \|_X$, $\| : \|_C$ and so on. We assume the following hypotheses.

(A) There exist positive constants $M', \omega$ such that

$$\|U(t, s)\| \leq M'e^{\omega(t-s)}, \quad 0 \leq s \leq t \leq T.$$ 

Here, we put $M = M'e^{\omega T}$.

(F1) The nonlinear function $F$ is defined on $[0, T] \times C \times L^2(0, T : U)$ and is uniformly Lipschitz on $x$ and $u$:

$$\|F(t, x, u) - F(t, y, v)\| \leq L_1\|x - y\|_C + L_2\|u - v\|_{L^2(0, T : U)}$$

for $x, y \in C$ and $u, v \in L^2(0, T : U)$.

We consider the nonlinear system

$$\dot{x}(\phi) = A(t)x(\phi) + F(t, x(t, \phi ; u), u(t)) + (Bu)(t),$$

where the linear operator $A(t)$ generate a strongly continuous evolution system $\{U(t, s)\}$ on $X$ and is continuously initially observable, there a unique mild solution is given as, for each $u$ in $L^2(0, T : U)$,

$$x_t(\phi ; u)(0)$$

$$= U(t, 0)\phi(0) + \int_0^t U(t, s)\{F(s, x_s(\phi ; u), u(s) + (Bu)(s)\}ds.$$ 

The solution mapping $W$ from $L^2(0, T : U)$ to $C(0, T : C)$ can be defined by

$$W(u)(t) = x_t(\phi ; u)(\cdot),$$

And assume the solution mapping is completely continuous.
THEOREM 1. Let \( u(\cdot) \in U \) and \( \phi \in C \). Then under Hypothesis \((F1)\) the solution mapping \((Wu)(t) = x_t(\phi : u)(\cdot)\) of \((2)\) satisfies

\[
\|x_t(\phi : u)\|_C \leq (M\|\phi\|_C + ML_2\sqrt{T}\|u\| + M\sqrt{T}\|B\|\|u\|)\exp(L_1MT)
\]

where \( L_1, L_2 \) and \( M \) are constants for \( 0 \leq t \leq T \).

Proof.

\[
\|x_{t+\theta}(\phi : u)(0)\|_X \\
\leq M\|\phi(0)\|_X + M\int_0^{t+\theta} \{\|F(s, x_s(\phi : u), u(s))\|_X + \|B\|\|u\|\} ds \\
\leq M\|\phi(0)\|_X + M\int_0^{t+\theta} \{L_1\|x_s(\phi : u)\|_C + L_2\|u\|\} ds \\
\quad + M\|B\|\|u\|\sqrt{t+\theta} - h \leq \theta \leq 0 \\
= M\|\phi(0)\|_X + ML_1\int_0^{t+\theta} \|x_s(\phi : u)\|_C ds + ML_2\|u\|\sqrt{t+\theta} \\
\quad + M\|B\|\|u\|\sqrt{t+\theta}.
\]

Hence

\[
\sup_{-h \leq \theta \leq 0} \|x_t(\phi : u)(\theta)\|_X \leq M\|\phi\|_C + ML_1\int_0^t \|x_s(\phi : u)\|_C ds \\
\quad + ML_2\|u\|\sqrt{t} + M\|B\|\|u\|\sqrt{t}.
\]

Thus we have

\[
\|x_t(\phi : u)\|_C \leq M\|\phi\|_C + ML_2\|u\|\sqrt{t} + M\|B\|\|u\|\sqrt{t} \\
\quad + ML_1\int_0^t \|x_s(\phi : u)\|_C ds.
\]

By Gronwall's inequality,

\[
\|x_t(\phi : u)\|_C \leq (M\|\phi\|_C + ML_2\|u\|\sqrt{T} + M\|B\|\|u\|\sqrt{T})\exp(L_1MT).
\]
THEOREM 2. Let \( u_1(\cdot) \) and \( u_2(\cdot) \) be in \( U \). Then under hypothesis (F1) the solution mapping \((Wu)(t) = x_t(\phi : u)\) of (2) satisfies

\[
\|x_t(\phi : u_1)(\cdot) - x_t(\phi : u_2)(\cdot)\|_C \\
\leq \{(L_2 + \|B\|)M\sqrt{T}\|u_1(\cdot) - u_2(\cdot)\|_{L^2(0,T;X)}\} \exp(ML_1T).
\]

Proof. From hypotheses and system (2) we have, for \(-h \leq \theta \leq 0\),

\[
\|x_t(\phi : u_1)(\theta) - x_t(\phi : u_2)(\theta)\|_X \\
\leq M \int_0^{t+\theta} \|F(s, x_s(\phi : u_1), u_2(s)) - F(s, x_s(\phi : u_2), u_2(s))\| \\
+ \|Bu_1(s) - Bu_2(s)\|_{L^2(0,T;X)}\|ds \\
\leq ML_1 \int_0^{t+\theta} \|x_s(\phi : u_1) - x_s(\phi : u_2)\|_C\|ds \\
+ ML_2 \int_0^{t+\theta} \|u_1(s) - u_2(s)\|_{L^2(0,T;X)}\|ds \\
+ M\|B\| \int_0^{t+\theta} \|u_1(s) - u_2(s)\|_{L^2(0,T;X)}\|ds.
\]

Hence

\[
\sup_{-h \leq \theta \leq 0} \|x_t(\phi : u_1) - x_t(\phi : u_2)(\theta)\|_X \\
= \|x_t(\phi : u_1) - x_t(\phi : u_2)\|_C \\
\leq ML_2 \sqrt{t} \|u_1 - u_2\| + M\|B\| \sqrt{t} \|u_1 - u_2\| \\
+ ML_1 \int_0^{t} \|x_s(\phi : u_1) - x_s(\phi : u_2)\|_C\|ds.
\]

By Gronwall's inequality,

\[
\|x_t(\phi : u_1) - x_t(\phi : u_2)\|_C \\
\leq \{(L_2 + \|B\|)M\sqrt{T}\|u_1(\cdot) - u_2(\cdot)\|_{L^2(0,T;X)}\} \exp(ML_1T).
\]
3. General Conclusions

In this section, we are going to give some general conclusions on both approximate controllability and exact reachability. Firstly some definitions are introduced.

**Definition 1.** The nonempty subset $K(F)$ in $C(0, T : X)$ by

$$K(F) = \{x_t(\phi : u)(0) \in C(0, T : X) : x_t(\phi : u)(0)$$

$$= U(t, 0)\phi(0) + \int_0^t U(t, s)\{F(s, x_s(\phi : u), u(s)) + (Bu)(s)\} ds \quad u \in L^2(0, T : U)\}. \tag{4}$$

**Definition 2.** The control system (1) is called approximately controllable on $[0, T]$ if

$$\overline{K(F)} = X.$$

**Definition 3.** For each $h \in X$ define

$$V_{(0, T)}[h] = \{u(.)|u(.) \in L^2(0, T : U) \quad \text{with} \quad x_T(\phi : u) = h\}.$$

If $V_{(0, T)}[h] \neq \emptyset$ (empty set in $L^2(0, T; U)$), then the delay volterra control system (1) is called $h$-exactly reachable from the origin on $[0, T]$.

While discussing approximate controllability and exact reachability for the delay volterra control system (1), we consider two families of associated quadratic optimal control problems

$$\begin{align*}
(Inf) \quad J_\epsilon(u; h) &= \|x_T(\phi : u) - h\|^2 + \epsilon\|u(.)\|^2_{L^2(0, T; U)} \\
\text{for} \; \epsilon > 0, \quad \text{and} \\
(Inf) \quad I_\epsilon(u; h) &= \frac{1}{\epsilon}\|x_T(\phi : u) - h\|^2 + \|u(.)\|^2_{L^2(0, T; U)}
\end{align*} \tag{5}$$
for $\epsilon > 0$, where $x_T(\phi : u)$ is the terminal state of the system (1) at time $T$.

For my given $h \in X$ and $\epsilon > 0$ there exists some control $u_\epsilon(\cdot) \in L^2(0, T; U)$ such that

\begin{equation}
J_\epsilon(u_\epsilon : h) = \inf_{u(\cdot) \in L^2(0, T; U)} J_\epsilon(u ; h).
\end{equation}

and

\begin{equation}
I_\epsilon(u_\epsilon : h) = \inf_{u(\cdot) \in L^2(0, T; U)} I_\epsilon(u : h).
\end{equation}

The control $u(\cdot)$ is called minimization element of the nonlinear functions $J_\epsilon(u : h)$ and $I_\epsilon(u : h)$.

**Theorem 3.** Assume $h \in X$. Then $h$ is in $\overline{K(F)}$ if and only if

\begin{equation}
\lim_{\epsilon \to 0} J_\epsilon(u_\epsilon : h) = 0.
\end{equation}

**Proof.** Let $h$ be an arbitrary element in $\overline{K(F)}$. Then for any given integer $N > 0$ there exists some control $v_N(\cdot) \in L^2(0, T ; U)$ such that

\[ \|x_T(\phi : v_N) - h\| < \frac{1}{N}, \quad N = 1, 2, \ldots. \]

Thus

\[ \lim_{\epsilon \to 0} J_\epsilon(u_\epsilon : h) \leq \lim_{\epsilon \to 0} J_\epsilon(v_N : h) \]

\[ \leq \lim_{\epsilon \to 0} \left( \frac{1}{N^2} + \epsilon \|v_N(\cdot)\|^2_{L^2(0, T ; U)} \right) = \frac{1}{N^2}. \]

Taking $N \to \infty$ in above we obtain (9).

Conversely, if (9) holds for some $h \in X$, then

\[ \lim_{\epsilon \to 0} \|x_T(\phi : u_\epsilon) - h\|^2 \leq \lim_{\epsilon \to 0} J_\epsilon(u_\epsilon : h) = 0, \]

and, equivalently, $h \in \overline{K(F)}$. 
Corollary 1. The system (1) is approximately controllable if and only if (9) holds for every $h \in X$.

Proof. Follows directly from Theorem 3.

Theorem 4. The system (1) is $h$-exactly reachable if and only if $I_{\varepsilon}(u_{\varepsilon} : h)$ is uniformly bounded for $0 < \varepsilon < \infty$.

Proof. Suppose the abstract control system (1) is $h$-exactly reachable and $v(\cdot)$ is an arbitrary control in $V_{(0,T)}[h]$. Then for any $\varepsilon > 0$

$$I_{\varepsilon}(u_{\varepsilon} : h) \leq I_{\varepsilon}(v : h) = \|v(\cdot)\|_{L^2(0,T;U)}^2$$

On the other hand, if $I_{\varepsilon}(u_{\varepsilon} : h)$ is uniformly bounded for $0 < \varepsilon < \infty$ holds for some $h \in X$, then

$$\lim_{\varepsilon \to 0} J_{\varepsilon}(u_{\varepsilon}) = \lim_{\varepsilon \to 0} \varepsilon I_{\varepsilon}(u_{\varepsilon} ; h) = 0.$$ 

Moreover, there exists some constant $M(h)$ independent of $\varepsilon > 0$ such that

$$\|u_{\varepsilon}(\cdot)\|_{L^2(0,T;U)}^2 \leq I_{\varepsilon}(u_{\varepsilon} ; h) \leq M(h)$$

Thus there exists some monotone sequence $\{\varepsilon_n ; n = 1, 2, \cdots \}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $w - \lim_{n \to \infty} u_{\varepsilon_n}(\cdot) = u^*(\cdot)$ in $L^2(0,T : U)$. Hence

$$\|x_T(\phi : u^*) - h\|^2 \leq \lim_{n \to \infty} \|x_T(\phi : u_{\varepsilon_n}) - h\|^2 \leq \lim_{n \to \infty} J_{\varepsilon_n}(u_{\varepsilon_n} ; h) = 0.$$ 

Thus, $u^*(\cdot) \in V_{(0,T)}[h]$ and the system (1) is $h$-exactly reachable.
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