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CONTROLLABILITY PROPERTIES OF DELAY VOLTERRA CONTROL SYSTEM

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1. Introduction

We consider the following delay volterra control system

$$
(1)\qquad \begin{aligned} x_t(\phi : u)(0) &= U(t,0)\phi(0) \\ &\quad + \int_0^t U(t,s)\{F(s, x_s(\phi : u), u(s)) + Bu(s)\}ds \\ x_0(\theta) &= \phi \in C. \end{aligned}
$$

Here, let X and U be Hilbert spaces. The state function $x(t)$, $0 \leq t \leq T$, takes values in X and the control function u is given in $L^2(0, T : U)$ and $U(t, s)$ is a linear evolution operator on X. Let C be a Banach space of all continuous functions from an interval of the form $I = [-h, 0]$ to X with the norm defined by supremum. If a function u is continuous from $I \cup [0, T]$ to X, then u_t is an element in C which has point-wise definition $u_t(\theta) = u(t + \theta)$ for $\theta \in I$.

We assume that F is a nonlinear function from $[0, T] \times C \times L^2(0, T : U)$ to X and B is a bounded linear operator from $L^2(0, T : U)$ to $L^2(0, T : X)$.

The purpose of this paper is to give some general conclusions on both approximate controllability and exact reachability.

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2. Preliminaries and Estimation

The norm of the space $L^2(0,T:X)$ or $L^2(0,T:U)$ is denoted by $\|\cdot\|_X$, $\|\cdot\|_C$ and so on. We assume the following hypotheses.

(A) There exist positive constants M', ω such that

$$\|U(t,s)\| \leq M'e^{\omega(t-s)}, \quad 0 \leq s \leq t \leq T.$$

Here, we put $M = M'e^{\omega T}$.

(F1) The nonlinear function F is defined on $[0,T] \times C \times L^2(0,T:U)$ and is uniformly Lipschitz on x and u:

$$\|F(t,x,u) - F(t,y,v)\| \leq L_1\|x-y\|_C + L_2\|u-v\|_{L^2(0,T:U)}$$

for $x, y \in C$ and $u, v \in L^2(0,T:U)$.

We consider the nonlinear system

$$\dot{x}_t(\phi) = A(t)x_t(\phi) + F(t, x_t(\phi:u), u(t)) + (Bu)(t),$$

where the linear operator $A(t)$ generate a strongly continuous evolution system $\{U(t,s)\}$ on X and is continuously initially observable, there a unique mild solution is given as, for each u in $L^2(0,T:U)$,

$$\begin{aligned} &x_t(\phi:u)(0) \\ (2) \quad &= U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi:u), u(s) + (Bu)(s)\}ds. \end{aligned}$$

The solution mapping W from $L^2(0.T:U)$ to $C(0,T:C)$ can be defined by

$$(3) \qquad (Wu)(t) = x_t(\phi:u)(\cdot),$$

And assume the solution mapping is completely continuous.

THEOREM 1. *Let $u(\cdot) \in U$ and $\phi \in C$. Then under Hypothesis $(F1)$ the solution mapping $(Wu)(t) = x_t(\phi : u)(\cdot)$ of (2) satisfies*

$$\|x_t(\phi : u)\|_C \leq (M\|\phi\|_C + ML_2\sqrt{T}\|u\| + M\sqrt{T}\|B\|\|u\|)\exp(L_1 MT)$$

where L_1, L_2 and M are constants for $0 \leq t \leq T$.

Proof.

$$\begin{aligned}
&\|x_{t+\theta}(\phi : u)(0)\|_X \\
\leq& M\|\phi(0)\|_X + M\int_0^{t+\theta} \{\|F(s, x_s(\phi : u), u(s)\|_X + \|B\|\|u\|_X\} ds \\
\leq& M\|\phi(0)\|_X + M\int_0^{t+\theta} \{L_1\|x_s(\phi : u)\|_C + L_2\|u\|\} ds \\
&\qquad + M\|B\|\|u\|\sqrt{t+\theta} \quad -h \leq \theta \leq 0 \\
=& M\|\phi(0)\|_X + ML_1\int_0^{t+\theta} \|x_s(\phi : u)\|_C ds + ML_2\|u\|\sqrt{t+\theta} \\
&\qquad + M\|B\|\|u\|\sqrt{t+\theta}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{-h\leq\theta\leq 0} \|x_t(\phi : u)(\theta)\|_X \leq& M\|\phi\|_C + ML_1\int_0^t \|x_s(\phi : u)\|_C ds \\
&+ ML_2\|u\|\sqrt{t} + M\|B\|\|u\|\sqrt{t}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_t(\phi : u)\|_C \leq& M\|\phi\|_C + ML_2\|u\|\sqrt{t} + M\|B\|\|u\|\sqrt{t} \\
&+ ML_1\int_0^t \|x_s(\phi : u)\|_C ds.
\end{aligned}$$

By Gronwall's inequality,

$$\|x_t(\phi : u)\|_C \leq (M\|\phi\|_C + ML_2\|u\|\sqrt{T} + M\|B\|\|u\|\sqrt{T})\exp(L_1 MT).$$

THEOREM 2. *Let $u_1(\cdot)$ and $u_2(\cdot)$ be in U. Then under hypothesis (F1) the solution mapping $(Wu)(t) = x_t(\phi : u)$ of (2) satisfies*

$$\begin{aligned}&\|x_t(\phi : u_1)(\cdot) - x_t(\phi : u_2)(\cdot)\|_C \\ \leq&\{(L_2 + \|B\|)M\sqrt{T}\|u_1(\cdot) - u_2(\cdot)\|_{L^2(0,T:X)}\} \exp(ML_1T).\end{aligned}$$

Proof. From hypotheses and system (2) we have, for $-h \leq \theta \leq 0$,

$$\begin{aligned}&\|x_t(\phi : u_1)(\theta) - x_t(\phi : u_2)(\theta)\|_X \\ \leq&M\int_0^{t+\theta}\{\|F(s, x_s(\phi : u_1), u_2(s)) - F(s, x_s(\phi : u_2), \bar{u}_2(s))\| \\ &\qquad + \|Bu_1(s) - Bu_2(s)\|_{L^2(0,T:X)}\}ds \\ \leq&ML_1\int_0^{t+\theta}\|x_s(\phi : u_1) - x_s(\phi : u_2)\|_C ds \\ &\quad + ML_2\int_0^{t+\theta}\|u_1(s) - u_2(s)\|_{L^2(0,T:X)}ds \\ &\quad + M\|B\|\int_0^{t+\theta}\|u_1(s) - u_2(s)\|_{L^2(0,T:X)}ds.\end{aligned}$$

Hence

$$\begin{aligned}&\sup_{-h\leq\theta\leq 0}\|x_t(\phi : u_1) - x_t(\phi : u_2)(\theta)\|_X \\ =&\|x_t(\phi : u_1) - x_t(\phi : u_2)\|_C \\ \leq&ML_2\sqrt{t}\|u_1 - u_2\| + M\|B\|\sqrt{t}\|u_1 - u_2\| \\ &\qquad + ML_1\int_0^t\|x_s(\phi : u_1) - x_s(\phi : u_2)\|_C ds.\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}&\|x_t(\phi : u_1) - x_t(\phi : u_2)\|_C \\ \leq&\{(L_2 + \|B\|)M\sqrt{T}\|u_1(\cdot) - u_2(\cdot)\|_{L^2(0,T:X)}\} \exp(ML_1T).\end{aligned}$$

3. General Conclusions

In this section, we are going to give some general conclusions on both approximate controllability and exact reachability. Firstly some definitions are introduced.

DEFINITION 1. The nonempty subset $K(F)$ in $C(0,T:X)$ by

$$
\begin{aligned}
K(F) = \{ & x_t(\phi : u)(0) \in C(0,T:X) \;:\; x_t(\phi : u)(0) \\
& = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi : u), u(s)) \\
& + (Bu)(s)\}ds \qquad u \in L^2(0,T:U)\}.
\end{aligned}
\tag{4}
$$

DEFINITION 2. The control system (1) is called approximately controllable on $[0,T]$ if

$$\overline{K(F)} = X.$$

DEFINITION 3. For each $h \in X$ define

$$V_{(0,T)}[h] = \{u(.)|u(.) \in L^2(0,T:U) \quad \text{with} \quad x_T(\phi : u) = h\}.$$

If $V_{(0,T)}[h] \neq \phi$ (empty set in $L^2(0,T;U)$), then the delay volterra control system (1) is called h-exactly reachable from the origin on $[0,T]$.

While discussing approximate controllability and exact reachability for the delay volterra control system (1), we consider two families of associated quadratic optimal control problems

$$(Inf) \quad J_\epsilon(u;h) = \|x_T(\phi;u) - h\|^2 + \epsilon\|u(.)\|^2_{L^2(0,T;U)} \tag{5}$$

for $\epsilon > 0$, and

$$(Inf) \quad I_\epsilon(u;h) = \frac{1}{\epsilon}\|x_T(\phi : u) - h\|^2 + \|u(.)\|^2_{L^2(0.T:U)} \tag{6}$$

for $\epsilon > 0$, where $x_T(\phi : u)$ is the terminal state of the system(1) at time T.

For my given $h \in X$ and $\epsilon > 0$ there exists some control $u_\epsilon(\ ) \in L^2(0,T;U)$ such that

$$J_\epsilon(u_\epsilon : h) = \inf_{u(\cdot)\in L^2(0,T:U)} J_\epsilon(u;h) \tag{7}$$

and

$$I_\epsilon(u_\epsilon : h) = \inf_{u(\cdot)\in L^2(0,T:U)} I_\epsilon(u : h). \tag{8}$$

The control $u(\cdot)$ is called minimization element of the nonlinear functions $J_\epsilon(u : h)$ and $I_\epsilon(u : h)$.

THEOREM 3. *Assume $h \in X$. Then h is in $\overline{K(F)}$ if and only if*

$$\lim_{\epsilon\to 0} J_\epsilon(u_\epsilon : h) = 0. \tag{9}$$

Proof. Let h be an arbitrary element in $\overline{K(F)}$. Then for any given integer $N > 0$ there exists some control $v_N(\cdot) \in L^2(0,T : U)$ such that

$$\|x_T(\phi : v_N) - h\| < \frac{1}{N}, \qquad N = 1, 2, \ldots.$$

Thus

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon : h) &\le \lim_{\varepsilon\to 0} J_\varepsilon(v_N : h)\\ &\le \lim_{\varepsilon\to 0}\Big(\frac{1}{N^2} + \varepsilon\|v_N(\cdot)\|^2_{L^2(0,T:U)}\Big) = \frac{1}{N^2}.\end{aligned}$$

Taking $N \to \infty$ in above we obtain (9).

Conversely, if (9) holds for some $h \in X$, then

$$\lim_{\varepsilon\to 0}\|x_T(\phi : u_\varepsilon) - h\|^2 \le \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon : h) = 0,$$

and, equivalently, $h \in \overline{K(F)}$.

COROLLARY 1. *The system (1) is approximately controllable if and only if (9) holds for every $h \in X$.*

Proof. Follows directly from Theorem 3.

THEOREM 4. *The system (1) is h-exactly reachable if and only if $I_\varepsilon(u_\varepsilon : h)$ is uniformly bounded for $0 < \varepsilon < \infty$.*

Proof. Suppose the abstract control system (1) is h-exaetly reachable and $v(\cdot)$ is an arbitrary control in $V_{(0,T)}[h]$. Then for any $\epsilon > 0$

$$I_\epsilon(u_\epsilon : h) \le I_\epsilon(v : h) = \|v(\cdot)\|^2_{L^2(0,T:U)}$$

On the other hand, if $I_\epsilon(u_\epsilon : h)$ is uniformly bounded for $0 < \epsilon < \infty$ holds for some $h \in X$, then

$$\lim_{\epsilon \to 0} J_\epsilon(u_\epsilon) = \lim_{\epsilon \to 0} \epsilon I_\epsilon(u_\epsilon; h) = 0.$$

Moreover, there exists some constant $M(h)$ independent of $\epsilon > 0$ such that

$$\|u_\epsilon(\cdot)\|^2_{L^2(0,T,U)} \le I_\epsilon(u_\epsilon; h) \le M(h)$$

Thus there exists some monotone sequence $\{\epsilon_n; n = 1, 2, \cdots\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that $w - \lim_{n \to \infty} u_{\epsilon_n}(\cdot) = u^*(\cdot)$ in $L^2(0, T : U)$. Hence

$$\|x_T(\phi : u^*) - h\|^2 \le \varliminf_{n \to \infty} \|x_T(\phi : u_{\epsilon_n}) - h\|^2 \le \lim_{n \to \infty} J_{\epsilon_n}(u_{\epsilon_n} : h) = 0.$$

Thus, $u^*(\cdot) \in V_{(0,T)}[h]$ and the system (1) is h-exactly reachable.

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