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ON A PROBLEM OF G–PART OF BCI–ALGEBRAS

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In this note, we first give a positive answer of the following open problem in [5]:

Does the inverse of [5; Theorem 10] hold?

Next, for any subalgebra S of a BCI-algebra X, we obtain a number of statements, each of which is equivalent to that

$$L(S) = \{x \in S : x = 0 * (0 * x)\}$$

is an ideal of X. Finally, we give some of other characterizations of KL-product BCI-algebras as a complement of [6] and [8].

The set $L(X)$ of all atoms in a BCI-algebra X is a p-semisimple subalgebra of X; hence it is said to be p-semisimple part of X. But it, in general, may not be an ideal of X. W. P. Huang [2] and J. Meng and X. L. Xin [8] considered the question that in order that $L(X)$ is an ideal of X, what condition does X satisfy? To be motivated by [2], Y. B. Jun and E. H. Roh [5] investigated the G-part of a BCI-algebra X and proved the following.

"THEOREM 10". *If S is a subalgebra of X and $G(S) = \{x \in S : 0 * x = x\}$ an ideal of X, then for any $x, y \in B(X)$ and for any $a, b \in G(S)$,*

$$x * a = y * b \text{ implies } x = y \text{ and } a = b.$$

In [5], they posed the open problem:

(JR) Does the inverse of "Theorem 10" hold?

In this note, one of our mainly aims is to give a positive answer to this problem. Following the idea of [5] we will also discuss that for a subalgebra S of X, what is the condition under which

$$L(S) = \{x \in S : x = 0 * (0 * x)\}$$

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is an ideal of X?

Throughout this paper, X will always mean a BCI-algebra without further explanation. We need to review some definitions and results for the development of this paper.

By a BCI-algebra we mean an abstract algebra $(X;*,0)$ of type (2, 0) satisfying the following conditions:

BCI-1 $((x*y)*(x*z))*(z*y)=0$;

BCI-2 $(x*(x*y))*y=0$;

BCI-3 $x*x=0$;

BCI-4 $x*y=0$ and $y*x=0$ imply $x=y$.

A BCI-algebra X satisfying

BCK-5 $0*x=0$ for all x in X

is said to be a BCK-algebra.

In a BCI-algebra X we can define an ordering relation $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

For a BCI-algebra X we have

(1) $x*0=x$,

(2) $(x*y)*z=(x*z)*y$,

(3) $((x*z)*(y*z))*(x*y)=0$,

(4) $0*(x*y)=(0*x)*(0*y)$.

In this note, we would use these results at several different occasions, however, we would not mention them explicity.

A BCI-algebra X is said to be associative ([1]) if it satisfies

(5) $(x*y)*z=x*(y*z)$.

In an associative BCI-algebra, the following identities hold:

(6) $0*x=x$,

(7) $x*y=y*x$.

The set $B(X)=\{x\in X:0*x=0\}$ is called the BCK-part of X; clearly, $0\in B(X)$ and $(B(X);*,0)$ is a BCK-subalgebra of X. In general, $B(X)\neq\{0\}$; if $B(X)=\{0\}$, then X is said to be p-semisimple([10]). In our joint paper [7], we investigated atoms in a BCI-algebra.

Definition 1 ([7]). An element a of X is called to be an atom of X if, for any $x\in X$,

(8) $x*a=0$ implies $x=a$.

The set of all the atoms is denoted by $L(X)$, which is also called the p-semisimple part of X. For any $a \in L(X)$, the set

$$V(a) = \{x \in X : a * x = 0\}$$

is said to be a branch of X.

We will need the following(see [7] and [10]):

(9) $a \in X$ is an atom iff $a = x * (x * a)$ for any x in X;

(10) $L(X)$ is a subalgebra of X, that is, $a, b \in L(X)$ imply $a * b \in L(X)$;

(11) If $a, b \in L(X)$, then for any $x \in V(a)$ and $y \in V(b)$, we have $x * y \in V(a * b)$;

(12) If x, y belong to the same branch, then $x * y \in B(X)$;

(13) For any $x \in V(a)$ and any $b \in L(X)$, $b * x = b * a$;

(14) For all $x \in X$, $0 * x \in L(X)$.

Definition 2 ([3]). A nonempty subset I of X is called an ideal if it satisfies

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

The set of all the ideals of X is denoted by $\mathcal{I}(X)$. The set of all subalgebras of X is denoted by $Sub(X)$. In general, a subalgebra need not be an ideal. But T. D. Lei and C. C. Xi proved

Lemma 3 ([10]). *Suppose X is a p-semisimple BCI-algebra, then $Sub(X) \subseteq \mathcal{I}(X)$.*

Y. B. Jun and E. H. Roh [5] investigated the G-part of X.

Definition 4 ([5]). For any subset S of X, define

$$G(S) = \{x \in S : 0 * x = x\}.$$

In particular, if $S = X$ then we call $G(X)$ the G-part of X.

Lemma 5 ([5]). *If $S \in Sub(X)$ then $G(S) \in Sub(X)$.*

The following corollary is obvious.

COROLLARY 6. *If $S \in Sub(X)$, then $G(S)$ is an associative subalgebra of $L(X)$, in particular, $G(X)$ is an associative subalgebra of $L(X)$.*

Now we have all the background needed to solve the problem (JR).

THEOREM 7. *Let $S \in Sub(X)$. Then the following are equivalent:*
(15) $G(S) \in \mathcal{I}(X)$;
(16) *for any $x, y \in B(X)$ and for any $a, b \in G(S)$*

$$x * a = y * b \text{ implies } x = y \text{ and } a = b;$$

(17) *for any $x, y \in B(X)$ and for any $a \in G(S)$*

$$x * a = y * a \textit{ implies } x = y;$$

(18) *for any $x \in B(X)$ and any $a \in G(S)$*

$$x * a = 0 * a \textit{ implies } x = 0.$$

Proof. (15) $\Rightarrow$ (16). See [5; Theorem 10].
(16) $\Rightarrow$ (17) $\Rightarrow$ (18) are trivial.
(18) $\Rightarrow$ (15). Assume $x * b \in G(S)$ and $b \in G(S)$. Denote $a = 0 * (0 * x)$, then $a \in L(X)$. By (11) we have $x * b \in V(a * b)$, that is, $a * b \leq x * b \in G(S)$. By (8), $x * b = a * b$. Thus

$$(x * a) * b = (x * b) * a = (a * b) * a = (a * a) * b = 0 * b.$$

Observe $x * a \in B(X)$ by (12), then using (18) we have $x * a = 0$, and so $x = a$ by (8). Hence $x * b \in G(S)$, $b \in G(S)$ and $x \in L(X)$. By combining Corollary 6 and Lemma 3 we know that $G(S)$ is an ideal of $L(X)$, it follows that $x \in G(S)$. This says that $G(S) \in \mathcal{I}(X)$, proving the theorem.

The implication (16) $\Rightarrow$ (15) gives a positive answer of the problem (JR). Below we will give further results.

THEOREM 8. *If $S \in Sub(X)$, then the following are equivalent:*
(15) $G(S) \in \mathcal{I}(X)$,
(19) *for any $x, y \in X$ and for any $a \in G(S)$,*

$$x * a = y * a \text{ implies } x = y,$$

(20) *for any $x \in X$ and for any $a, b \in G(S)$,*

$$x * a = b * a \text{ implies } x = b,$$

(21) *for any $x \in X$ and for any $a \in G(S)$*

$$x * a = 0 * a \text{ implies } x = 0.$$

Proof. (15) $\Rightarrow$ (19). Suppose $G(S) \in \mathcal{I}(X)$ and $x * a = y * a$ where $x, y \in X$ and $a \in G(S)$, then

$$(x * y) * a = (x * a) * y = (y * a) * y = (y * y) * a = 0 * a \in G(S),$$

and so $x * y \in G(S)$ by (15). Hence by (9) and (7)

$$x * y = a * (a * (x * y)) = a * ((x * y) * a) = a * (0 * a) = a * a = 0.$$

In the same argument, we have $y * x = 0$. Therefore $x = y$, (19) holds.
(19) $\Rightarrow$ (20) $\Rightarrow$ (21) are trivial.
(21) $\Rightarrow$ (15). Obviously, (21) $\Rightarrow$ (18). Combining Theorem 7 we know that (15) is true. The proof is complete.

THEOREM 9. *For a subalgebra S of X, $G(S) \in \mathcal{I}(X)$ if and only if, for any $x \in X$ and for any $b \in G(S)$*
(22) $x = (x * b) * (0 * b)$.

Proof. Suppose $G(S) \in \mathcal{I}(X)$ and $b \in G(S)$. For any $x \in X$, by (14) and (9) we have

$$(x * ((x * b) * (0 * b))) * b = (x * b) * ((x * b) * (0 * b)) = 0 * b,$$

hence by (21)

$$x * ((x * b) * (0 * b)) = 0.$$

On the other hand,

$$((x * b) * (0 * b)) * x = ((x * x) * b) * (0 * b) = (0 * b) * (0 * b) = 0.$$

Thus $x = (x * b) * (0 * b)$, namely, (22) holds.

Conversely, suppose (22) holds and $x * b \in G(S)$, $b \in G(S)$. Observe $0 * b \in G(S)$, we have

$$x = (x * b) * (0 * b) \in G(S),$$

which says $G(S) \in \mathcal{I}(X)$. The proof is complete.

THEOREM 10. *Suppose $S \in Sub(X)$. Then $G(S) \in \mathcal{I}(X)$ if and only if, for any $x, y \in X$ and for any $a, b \in G(S)$,*

(23) $(x * a) * (y * b) = (x * y) * (a * b)$.

Proof. Suppose $G(S) \in \mathcal{I}(X)$. Let $a, b \in G(S)$. Then for any $x, y \in X$

$$\begin{aligned}
&(((x * y) * (a * b)) * ((x * a) * (y * b))) * a \\
&= (((x * a) * (a * b)) * ((x * a) * (y * b))) * y \\
&\leq ((y * b) * (a * b)) * y \\
&\leq (y * a) * y \\
&= 0 * a,
\end{aligned}$$

and by (8),

$$(((x * y) * (a * b)) * ((x * a) * (y * b))) * a = 0 * a.$$

Using (21) we obtain

(24) $((x * y) * (a * b)) * ((x * a) * (y * b)) = 0.$

On the other hand,

$$\begin{aligned}
&(((x * a) * (y * b)) * ((x * y) * (a * b))) * (a * b) \\
&= (((x * (a * b)) * (y * b)) * ((x * y) * (a * b))) * a \\
&= (((x * (a * b)) * ((x * y) * (a * b))) * (y * b)) * a \\
&\leq ((x * (x * y)) * (y * b)) * a \\
&\leq (y * (y * b)) * a \\
&\leq b * a \\
&= 0 * (a * b);
\end{aligned}$$

hence

$$(((x * a) * (y * b)) * ((x * y) * (a * b))) * (a * b) = 0 * (a * b).$$

By (21) we have

(25) $((x * a) * (y * b)) * ((x * y) * (a * b)) = 0.$

Combining (24) and (25) we obtain

$$(x * a) * (y * b) = (x * y) * (a * b),$$

(23) holds.

Conversely, suppose (23) holds. If $x * a = y * a$ where $x, y \in X$ and $a \in G(S)$, then by (23)

$$x * y = (x * y) * (a * a) = (x * a) * (y * a) = 0.$$

Likewise we have $y * x = 0$, and so $x = y$. This shows that (19) holds. By Theorem 8, $G(S) \in \mathcal{I}(X)$. The proof is complete.

Observe that if X is quasi-associative([11]), then $G(X) = L(X)$, hence from Theorems 7–10 we have

COROLLARY 11. *If X is a quasi-associative BCI-algebra, then the following are equivalent:*

(26) $L(X) \in \mathcal{I}(X)$,

(27) *for any $x, y \in B(X)$ and for any $a, b \in L(X)$*

$$x * a = y * b \text{ implies } x = y \text{ and } a = b,$$

(28) *for any $x, y \in B(X)$ and for any $a \in L(X)$*

$$x * a = y * a \text{ implies } x = y,$$

(29) *for any $x \in B(X)$ and any $a \in L(X)$*

$$x * a = 0 * a \text{ implies } x = 0,$$

(30) *for any $x, y \in X$ and for any $a \in L(X)$*

$$x * a = y * a \text{ implies } x = y,$$

(31) *for any* $x \in X$ *and for any* $a, b \in L(X)$

$$x * a = b * a \text{ implies } x = b,$$

(32) *for any* $x \in X$ *and for any* $a \in L(X)$

$$x * a = 0 * a \text{ implies } x = 0,$$

(33) *for any* $x \in X$ *and for any* $a \in L(X)$

$$x = (x * a) * (0 * a),$$

(34) *for any* $x, y \in X$ *and for any* $a, b \in L(X)$

$$(x * a) * (y * b) = (x * y) * (a * b).$$

For a subset S of X, denote

$$L(S) = \{x \in S : x = 0 * (0 * x)\};$$

in particular, when $S = X$, $L(X)$ is precisely the set of all atoms of X. If $S \in Sub(X)$ then $L(S) \in Sub(L(X))$; if $L(S) \in \mathcal{I}(X)$ then $L(S) \in \mathcal{I}(L(X))$. To be motivated by Theorem 7, a natural question arises: does the similar results for $L(S)$ hold? In what follows we respond this question.

THEOREM 12. *Let* $S \in Sub(X)$. *Then the following are equivalent:*
(35) $L(S) \in \mathcal{I}(X)$,
(36) *for any* $x \in X$ *and for any* $a, b \in L(S)$

$$x * b = a * b \text{ implies } x = a,$$

(37) *for any* $x \in X$ *and* $a \in L(S)$

$$x * a = 0 * a \text{ implies } x = 0,$$

(38) *for any* $x, y \in X$ *and for any* $a \in L(S)$

$$x * a = y * a \text{ implies } x = y,$$

(39) *for any* $x, y \in B(X)$ *and for any* $a, b \in L(S)$

$$x * a = y * b \textit{ implies } x = y \text{ and } a = b,$$

(40) *for any* $x, y \in B(X)$ *and for any* $a \in L(S)$

$$x * a = y * a \textit{ implies } x = y,$$

(41) *for any* $x \in B(X)$ *and for any* $a \in L(S)$

$$x * a = 0 * a \textit{ implies } x = 0.$$

Proof. (35) $\Rightarrow$ (36). Suppose $L(S) \in \mathcal{I}(X)$. If $a, b \in L(S)$ then $a * b \in L(S)$ as $L(S) \in Sub(X)$. Hence for any $x \in X$, $x * b = a * b$ implies $x * b \in L(S)$, and furthermore, $x \in L(S)$. Thus by (9) and (13),

$$x = b * (b * x) = b * (0 * (x * b)) = b * (0 * (a * b)) = b * (b * a) = a,$$

namely, (36) holds.

(36) $\Rightarrow$ (37). It is immediate as $0 \in L(S)$.

(37) $\Rightarrow$ (38). Suppose (37) holds and let $x * a = y * a$ where $x, y \in X$ and $a \in L(S)$, then

$$(x * y) * a = (y * a) * y = 0 * a.$$

By (37), $x * y = 0$. Likewise for $y * x = 0$. Hence $x = y$. (38) is true.

(38) $\Rightarrow$ (39). Let $x * a = y * b$ where $x, y \in B(X)$ and $a, b \in L(S)$. Clearly, $0 * x = 0 * y = 0$. By (9) and (4)

$$a = 0*(0*a) = (0*x)*(0*a) = 0*(x*a) = (0*y)*(0*b) = 0*(0*b) = b.$$

Thus $x * a = y * a$ where $a \in L(S)$. (39) follows from (38).

(39) $\Rightarrow$ (40) $\Rightarrow$ (41) are trivial.

(41) $\Rightarrow$ (35). Suppose $x * a \in L(S)$ and $a \in L(S)$. By (11) $x * a = b * a$ where $b = 0 * (0 * x) \in L(X)$. Hence $(x * b) * a = 0 * a$. Since $x * b \in B(X)$ by (12), we have $x * b = 0$ by (41), and so $x = b$. This says $x \in L(X)$. Observe that $L(S) \in Sub(L(X))$, hence by Lemma 3 we have $L(S) \in \mathcal{I}(L(X))$. Thus $x * a \in L(S)$ and $a \in L(S)$ imply $x \in L(S)$ since $x \in L(X)$, that is, $L(S) \in \mathcal{I}(X)$. The proof is complete.

DEFINITION 13 ([8]). A BCI-algebra X is said to be of KL-product if there exist a BCK-algebra Y and a p-semisimple BCI-algebra Z such that $X \cong Y \times Z$.

In the setting of $S = X$, we put Theorem 12, [8; Theorems 1 and 3] and [6; Theorems 5, 6 and 7] together to obtain

THEOREM 14. *In a BCI-algebra X, the following are equivalent:*

(42) $L(X) \in \mathcal{I}(X)$,

(43) *for any $x \in X$ and for any $a, b \in L(X)$*

$$x * b = a * b \text{ implies } x = a,$$

(44) *for any $x \in X$ and $a \in L(X)$*

$$x * a = 0 * a \text{ implies } x = 0,$$

(45) *for any $x, y \in X$ and for any $a \in L(X)$*

$$x * a = y * a \text{ implies } x = y,$$

(46) *for any $x, y \in B(X)$ and for any $a, b \in L(X)$*

$$x * a = y * b \text{ implies } x = y \text{ and } a = b,$$

(47) *for any $x, y \in B(X)$ and for any $a \in L(X)$*

$$x * a = y * a \text{ implies } x = y,$$

(48) *for any $x \in B(X)$ and for any $a \in L(X)$*

$$x * a = 0 * a \text{ implies } x = 0,$$

(49) *X is of KL-product,*

(50) *for any $x \in X$ and for any $a \in L(X)$*

$$x = (x * a) * (0 * a),$$

(51) *for any $x, y \in X$ and for any $a, b \in L(X)$*

$$(x * a) * (y * b) = (x * y) * (a * b),$$

(52) *there exists an endomorphism f on X such that for any $a \in L(X)$, $f|_{V(a)}$, the restriction of f to $V(a)$, is a bijection from $V(a)$ onto $B(X)$.*

REMARK. The statement (42) $\Rightarrow$ (46) is precisely W. P. Huang [2; Theorem 1].

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