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ON A FIXED POINT THEOREM OF SOM–MUKHERJEE

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1. INTRODUCTION AND MAIN RESULTS

In [4] Som-Mukherjee gives the following definition.

DEFINITION 1. *Let X be an arbitrary set and Y any metric linear space. F is called a fuzzy mapping iff F is a mapping from the set X into $W(Y)$, where $W(Y)$ denotes the collection of all fuzzy sets A in Y such that (i) A_α is compact and convex in Y for each $\alpha \in [0.1]$ and (ii) $\sup_{y\in Y} A(y) = 1$, where $A_\alpha = \{x : A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \{x : A(\bar{x}) > 0\}$ the closure of $\{x : A(x) > 0\}$.*

DEFINITION 2. *Let (X,d) be a metric lincar space. A fuzzy mapping $F : X \to W(X)$ is nonexpansive if $D(F(x), F(y)) \leq d(x,y)$ for all $x, y \in X$.*

DEFINITION 3. *Let (X,d) be a metric space, $A, B \in W(X)$ and $\alpha \in [0,1]$, then we define*

$$p_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y)$$
$$D_\alpha(A,B) = H(A_\alpha, B_\alpha)$$

where H is the Hausdorff distance, and $D(A,B) = \sup_\alpha D_\alpha(A,B)$.

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LEMMA 4[3]. *Let (X,d) be a complete metric linear space, $F : X \to W(X)$ a fuzzy mapping and $x \in X$. Then there exists $\{u_x\} \in X$ such that $u_x \subset F(x)$.*

LEMMA 5. *Let X be a set, $x \in X, A \in W(X)$, and $\{x\}$ a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.*

LEMMA 6. *Let (X,d) be a metric space, then $p_\alpha(x, A) \leq d(x,y) + p_\alpha(y, A)$ for any $x, y \in X$.*

LEMMA 7. *Let (X,d) be a metric space. If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W(X)$*

The following Theorem due to Som-Mukherjee[3] is proved by the Lemmas 5, 6 and 7.

THEOREM 8. *Let X be a compact metric linear space. If $F : X \to W(X)$ is a nonexpansive mapping, then there exists a $x_0 \in X$ such that $x_0 \subset F(x_0)$.*

Proof. Choose a sequence $(x_n)_{n=1}^\infty$ in X such that $\{x_0\} \subset F(x_0)$ inductively. Then we can choose a convergent subsequence of $(x_n)_{n=1}^\infty$. Denote this subsequence $(x_n)_{n=1}^\infty$ again and $\lim_{n\to\infty} x_n = x_0$. So

$$\begin{aligned} P_\alpha(x_0, F(x_0)) &\leq d(x_0, x_{n+1}) + P_\alpha(x_{n+1}, F(x_0)) \\ &\leq d(x_0, x_{n+1}) + D_\alpha(F(x_n), F(x_0)) \\ &\leq d(x_0, x_{n+1}) + D(F(x_n), F(x_0)) \\ &\leq d(x_0, x_{n+1}) + d(x_n, x_0) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

for each $\alpha \in [0,1]$.

Hence $P_\alpha(x_0, F(x_0)) = 0$ for each $\alpha \in [0,1]$.

This implies that $\{x_0\} \subset F(x_0)$.

COROLLARY 5. *Let $(X, \|\cdot\|)$ be a compact normed vector space and $F : X \to W(X)$ a fuzzy mapping satisfying $D(F(x), F(y)) \leq \|x - y\|$. Then there exists a $z \in X$ such that $\{z\} \subset F(z)$.*

COROLLARY 6[1]. *Let X be a compact subset of a Banach space X. If F is a fuzzy mapping of K into $W(K)$ satisfying $D(F(x), F(y)) \leq \|x - y\|$, then there exists a point $z \in K$ such that $\{z\} \subset F(z)$.*

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