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GROUP MEMBERS IN NEAR–RINGS

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The elements of a near-ring together with the additive operation of the near-ring form a group. These elements together with the multipli cative operation of the near-ring,however, form a semigroup but may not form a group. It may occur that some subset G of a near-ring N under consideration does form a group under the multiplicative operation of the near-ring N. The set G is then a multiplicative group in the near-ring N. In this paper we consider only the multiplicative groups of a near-ring and never the additive groups. Thus we shall say G is a group in N, it being understood that the multiplicative operation of the near-ring N is the group operation of G. Accordingly, if a is an element of G and G is a group in a near-ring N, we shall say that a is a group member in N. In 1909, Arthur Ranum ([5]) introduced the notion of group membership and discussed it again in 1927 in [6]. The subject was also considered by H.K.Farahat and L.Mirsky ([3]) and W.E.Barnes and H.Schneider ([1]).

In this paper we obtained some properties of this notion.

DEFINITION 1. A near-ring is a system consisting of a set N and two binary operations in N called addition and multiplication such that (1). N together with addition is a group (2). N together with multiplication is a semigroup (3). The left distributive law holds.

DEFINITION 2. A set G is a group in a near-ring N if G is a subset of N and the elements of G form a group under the multiplicative operation of the near-ring N. If G is a group in a near-ring N and a is an element in G, then we say that a is a group member in N.

With these definitions we are now prepared to establish the first structure theorem for group membership in near-rings.

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THEOREM 3. *Let N be a near-ring and let a be a group member in N. Then there existsa group $M(a)$ in N such that every group in N containing a is a subgroup of $M(a)$. In particular, the identity element of any group containing a is the identity element of $M(a)$, and the inverse of a in any group to which it belongs is equal to its inverse in $M(a)$.*

Proof. Let $\{G_i\}_{i\in I}$ be the family of all groups in N which contain a, where I is an appropriate index set. For every i in I, denote by e_i the identity element of G_i and by a_i^{-1} the inverse of a in G_i. Then we have, for $i, j \in I$, $a_i^{-1} = a_i^{-1}e_i = a_i^{-1}a_i^{-1}a = a_i^{-1}a_i^{-1}ae_j = a_i^{-1}a_i^{-1}aaa_j^{-1} = a_i^{-1}e_iaa_j^{-1} = a_i^{-1}e_iaa_j^{-1} = a_i^{-1}aa_j^{-1} = a_i^{-1}ae_ja_j^{-1} = a_i^{-1}aaa_j^{-1}a_j^{-1} = e_iaa_j^{-1}a_j^{-1} = aa_j^{-1}a_j^{-1} = e_ja_j^{-1} = a_j^{-1}$. Therefore $e_i = aa_i = aa_j = e_j$. Thus all groups G_i,$i \in I$, have a common identity element, say e, and a common inverse,say a^{-1}, relative to e, in every group G_i. Now let $M(a)$ be the set of all elements x in N which can be represented by the form $x = x_1.x_2.x_3...x_k$, where x_i belongs to some G_i. It can be verified at once that $M(a)$ is a group with $(x_k)^{-1}...(x_1)^{-1}$ as the inverse of x and clearly $G_i, i \in I$, is a subgroup of $M(a)$.

DEFINITION 4. The group $M(a)$ of the above Theorem 3 is said to be the maximal group associated with a. The inverse of an element a, if it exists, will be denoted by a^{-1}.

THEOREM 5. *If N is a near-ring and a,b are group members in N, then $M(a)$ and $M(b)$ are either disjoint or identical.*

Proof. Let e be the identity of $M(a)$. It follows from the proof of Theorem 3 that $M(a) = M(e)$. Hence, if for some element c in N we have $c \in M(a)$ and $c \in M(b)$, then the identity of $M(b)$ is e. Thus we have $M(a) = M(e) = M(b)$.

DEFINITION 6. Let a be an element in a near-ring N. If there exists a positive integer n such that a^n is a group member in N, then a is said to have finite group index in N. The smallest such n is called the group index of a in N.

A near-ring N is said to have finite group index if every element in N has finite group index. If the group index of an element a is 1, then a^1 is a group member. Then from the definition, we have

THEOREM 7. *Let N be a near-ring and let a be an element in N with finite group index n. Then a is a group member in N if and only if $n = 1$.*

THEOREM 8. *Let an element a in a near-ring N have a finite group index n. Then a^t is a group member in N if and only if $t \geq n$. Furthermore, if $t = n$, then $M(a^t) = M(a^n)$.*

Proof. Let a^t be a group member in N. By the definition of the group index, t cannot be less than n. Hence we have $t \geq n$. Conversely, let $t \geq n$. If $t = n$, then $a^t = a^n$ and so a^t is a group member in N since a^n is a group member in N . Now suppose $t > n$. Denote the identity of $M(a^n)$ by e, and let $b = a^{-1}$. Then e is a two-sided identity for a^t, since $a^t e = (a^{t-n}a^n)e = a^{t-n}(a^n e) = a^{t-n}a^n = a^t$, and $ea^t = e(a^n a^{t-n}) = (ea^n)a^{t-n} = a^n a^{t-n} = a^t$. Next we show that a^t is invertible relative to e. Let p be an integer such that $pn > 2t$. Then $a^t(a^{pn-t}b^p) = (a^t a^{pn-t})b^p = a^{pn}b^p = (a^n b)^p = e^p = e$ and so $a^{pn-t}b^p$ is a right inverse for a^t. Since $(b^p a^{pn-t})a^t = b^p(a^{pn-t}a^t) = b^p a^{pn} = (ba^n)^p = e^p = e$, we have also that $b^p a^{pn-t}$ is a left inverse for a^t. But e is a two-sided identity for $a^{pn-t}b^p$,because $(a^{pn-t}b^p)e = (a^{pn-t}b^{p-1})(be) = (a^{pn-t}b^{p-1})b = a^{pn-t}b^p$ and $e(a^{pn-t}b^p) = (ea^t)(a^{pn-2t}b^p) = a^t(a^{pn-2t}b^p) = a^{pn-t}b^p$. Similarly, we may show that e is a two-sided identity for $b^p a^{pn-t}$. The uniqueness of the inverse of a^t follows from $a^{pn-t}b^p = e(a^{pn-t}b^p) = e^p(a^{pn-t}b^p) = (ba^n)^p(a^{pn-t}b^p) = (b^p a^{pn})(a^{pn-t}b^p) = (b^p a^{pn-t})(a^{pn}b^p)$ $=(b^p a^{pn-t})(a^n b)^p = (b^p a^{pn-t})e^p = (b^p a^{pn-t})e = b^p a^{pn-t}$. Thus a^t is invertible relative to e. Hence $a^t \in M(e)$ and so a^t is a group member in N.It follows that $M(a^t) = M(e) = M(a^n)$.

We now introduce the notion of pseudogroup.

DEFINITION 9. Let a be a group member in a near-ring N and let

$$P(a) = \{b |\ b \in N \text{ and } b^n \in M(a) \text{ for some positive integer } n\}.$$

The set $P(a)$ is called a pseudogroup in N. An element b in $P(a)$ is called a pseudogroup member in N.

THEOREM 10. . *If N is a near-ring and a, b are pseudogroup members in N, then the pseudogroups containing a and b are either disjoint or identical.*

Proof. It follows from the above Theorem 8 that each pseudogroup member in N is a member of one and only one pseudogroup.Hence, if for some element c in N we have that c is in the pseudogroup containing a and c is also in the pseudogroup containing b, then these pseudogroups are identical.

THEOREM 11. . *If N is a commutative near-ring,then every pseudogroup in N is a semigroup.*

Proof. Let a, b be members of some pseudogroup $P(e)P$ in N. Then there exist positive integers n and m such that $a^n \in M(e)$ and $b^m \in M(e)$. Hence $(ab)^{nm} = a^{nm}b^{nm} \in M(e)$, so that $P(e)$ is closed with respect to multiplication, and thus is a semigroup.

THEOREM 12. *Every finite near-ring consists entirely of pseudogroup members.*

Proof. Let N be a finite near-ring and let r be the number of elements in N. Let a any element in N. If a is nilpotent,then $a^n = 0$ for some positive integer n, and hence a is in the pseudogroup $P(0)$. Now suppose a is not nilpotent and form the sequence $a, a^2, ..., a^r, ...$ Since there are only r elements in N,including zero, not every element in the sequence is distinct. Let n be the smallest integer such that $a^n = a^{n+m}$ for some integer m.We now show that the set $G = \{a^n, a^{n+1}, ..., a^{n+m-1}\}$ is a group. We note first that G is a finite semigroup. Further, the cancellation laws hold in G, for if $a^{n+r}a^{n+s} = a^{n+r}a^{n+t}$, then $(n+r)+(n+s) = (n+r)+(n+t)(modulo m)$. It follows that $n+s = n+t(modulo m)$ so that $a^{n+s} = a^{n+t}$. The right cancellation may be shown similarly. But every finite semigroup in which the cancellation laws hold is a group. Hence G is a multiplicative group, so that a^n is a group member in N, and a is thus a pseudogroup in N.

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