GROUP MEMBERS IN NEAR-RINGS

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The elements of a near-ring together with the additive operation of the near-ring form a group. These elements together with the multiplicative operation of the near-ring, however, form a semigroup but may not form a group. It may occur that some subset $G$ of a near-ring $N$ under consideration does form a group under the multiplicative operation of the near-ring $N$. The set $G$ is then a multiplicative group in the near-ring $N$. In this paper we consider only the multiplicative groups of a near-ring and never the additive groups. Thus we shall say $G$ is a group in $N$, it being understood that the multiplicative operation of the near-ring $N$ is the group operation of $G$. Accordingly, if $a$ is an element of $G$ and $G$ is a group in a near-ring $N$, we shall say that $a$ is a group member in $N$. In 1909, Arthur Ranum ([5]) introduced the notion of group membership and discussed it again in 1927 in [6]. The subject was also considered by H.K.Farahat and L.Mirsky ([3]) and W.E.Barnes and H.Schneider ([1]).

In this paper we obtained some properties of this notion.

**Definition 1.** A near-ring is a system consisting of a set $N$ and two binary operations in $N$ called addition and multiplication such that (1). $N$ together with addition is a group (2). $N$ together with multiplication is a semigroup (3). The left distributive law holds.

**Definition 2.** A set $G$ is a group in a near-ring $N$ if $G$ is a subset of $N$ and the elements of $G$ form a group under the multiplicative operation of the near-ring $N$. If $G$ is a group in a near-ring $N$ and $a$ is an element in $G$, then we say that $a$ is a group member in $N$.

With these definitions we are now prepared to establish the first structure theorem for group membership in near-rings.

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Theorem 3. Let $N$ be a near-ring and let $a$ be a group member in $N$. Then there exists a group $M(a)$ in $N$ such that every group in $N$ containing $a$ is a subgroup of $M(a)$. In particular, the identity element of any group containing $a$ is the identity element of $M(a)$, and the inverse of $a$ in any group to which it belongs is equal to its inverse in $M(a)$.

Proof. Let $\{G_i\}_{i \in I}$ be the family of all groups in $N$ which contain $a$, where $I$ is an appropriate index set. For every $i$ in $I$, denote by $e_i$ the identity element of $G_i$ and by $a_i^{-1}$ the inverse of $a$ in $G_i$. Then we have, for $i,j \in I$, $a_i^{-1} = a_i^{-1}e_i = a_i^{-1}a_i^{-1}a = a_i^{-1}a_i^{-1}ae_i = a_i^{-1}a_i^{-1}aaa_j^{-1} = a_i^{-1}e_iaa_j^{-1} = a_i^{-1}e_iaa_j^{-1} = a_i^{-1}aa_j^{-1} = a_i^{-1}ae_ja_j^{-1} = a_i^{-1}aaa_j^{-1}a_j^{-1} = e_i aa_j^{-1}a_j^{-1} = e_i a_j^{-1} = a_j^{-1}$. Therefore $e_i = aa_i = aa_j = e_j$. Thus all groups $G_i, i \in I$, have a common identity element, say $e$, and a common inverse, say $a^{-1}$, relative to $e$, in every group $G_i$. Now let $M(a)$ be the set of all elements $x$ in $N$ which can be represented by the form $x = x_1 x_2 x_3 \ldots x_k$, where $x_i$ belongs to some $G_i$. It can be verified at once that $M(a)$ is a group with $(x_k)^{-1} \ldots (x_1)^{-1}$ as the inverse of $x$ and clearly $G_1, i \in I$, is a subgroup of $M(a)$.

Definition 4. The group $M(a)$ of the above Theorem 3 is said to be the maximal group associated with $a$. The inverse of an element $a$, if it exists, will be denoted by $a^{-1}$.

Theorem 5. If $N$ is a near-ring and $a, b$ are group members in $N$, then $M(a)$ and $M(b)$ are either disjoint or identical.

Proof. Let $e$ be the identity of $M(a)$. It follows from the proof of Theorem 3 that $M(a) = M(e)$. Hence, if for some element $c$ in $N$ we have $c \in M(a)$ and $c \in M(b)$, then the identity of $M(b)$ is $e$. Thus we have $M(a) = M(e) = M(b)$.

Definition 6. Let $a$ be an element in a near-ring $N$. If there exists a positive integer $n$ such that $a^n$ is a group member in $N$, then $a$ is said to have finite group index in $N$. The smallest such $n$ is called the group index of $a$ in $N$.

A near-ring $N$ is said to have finite group index if every element in $N$ has finite group index. If the group index of an element $a$ is 1, then $a^1$ is a group member. Then from the definition, we have
THEOREM 7. Let \( N \) be a near-ring and let \( a \) be an element in \( N \) with finite group index \( n \). Then \( a \) is a group member in \( N \) if and only if \( n = 1 \).

THEOREM 8. Let an element \( a \) in a near-ring \( N \) have a finite group index \( n \). Then \( a^t \) is a group member in \( N \) if and only if \( t \geq n \). Furthermore, if \( t = n \), then \( M(a^t) = M(a^n) \).

Proof. Let \( a^t \) be a group member in \( N \). By the definition of the group index, \( t \) cannot be less than \( n \). Hence we have \( t \geq n \). Conversely, let \( t \geq n \). If \( t = n \), then \( a^t = a^n \) and so \( a^t \) is a group member in \( N \) since \( a^n \) is a group member in \( N \). Now suppose \( t > n \). Denote the identity of \( M(a^n) \) by \( e \), and let \( b = a^{-1} \). Then \( e \) is a two-sided identity for \( a^t \), since \( a^t e = (a^t a^n) e = a^{t-n}(a^n e) = a^{t-n} a^n = a^t \), and \( ea^t = e(a^n a^{t-n}) = (ea^n)a^{t-n} = a^n a^{t-n} = a^t \). Next we show that \( a^t \) is invertible relative to \( e \). Let \( p \) be an integer such that \( pn > 2t \). Then \( a^t(a^{pn-t}b^p) = (a^t a^{pn-t})b^p = a^{pn}b^p = (a^nb)^p = e^p = e \) and so \( a^{pn-t}b^p \) is a right inverse for \( a^t \). Since \( (b^n a^{pn-t})a^t = b^p(a^{pn-t}a^t) = b^p a^{pn} = (ba^n)^p = e^p = e \), we have also that \( b^p a^{pn-t} \) is a left inverse for \( a^t \). But \( e \) is a two-sided identity for \( a^{pn-t}b^p \), because \( (a^{pn-t}b^p)e = (a^{pn-t}b^p-1)(be) = (a^{pn-t}b^p-1)b = a^{pn-t}b^p \) and \( e(a^{pn-t}b^p) = (ea^t)(a^{pn-2t}b^p) = a^t(a^{pn-2t}b^p) = a^{pn-t}b^p \). Similarly, we may show that \( e \) is a two-sided identity for \( b^p a^{pn-t} \). The uniqueness of the inverse of \( a^t \) follows from \( a^{pn-t}b^p = e(a^{pn-t}b^p)^t = e^p(a^{pn-t}b^p) = (ba^n)^p(a^{pn-t}b^p) = (b^p a^{pn-t})(a^{pn-t}b^p) = (b^p a^{pn-t})(a^{pn-t})e = b^p a^{pn-t} \). Thus \( a^t \) is invertible relative to \( e \). Hence \( a^t \in M(e) \) and so \( a^t \) is a group member in \( N \). It follows that \( M(a^t) = M(e) = M(a^n) \).

We now introduce the notion of pseudogroup.

DEFINITION 9. Let \( a \) be a group member in a near-ring \( N \) and let
\[
P(a) = \{ b \mid b \in N \text{ and } b^n \in M(a) \text{ for some positive integer } n \}.
\]
The set \( P(a) \) is called a pseudogroup in \( N \). An element \( b \) in \( P(a) \) is called a pseudogroup member in \( N \).

THEOREM 10. If \( N \) is a near-ring and \( a, b \) are pseudogroup members in \( N \), then the pseudogroups containing \( a \) and \( b \) are either disjoint or identical.
Proof. It follows from the above Theorem 8 that each pseudogroup member in $N$ is a member of one and only one pseudogroup. Hence, if for some element $c$ in $N$ we have that $c$ is in the pseudogroup containing $a$ and $c$ is also in the pseudogroup containing $b$, then these pseudogroups are identical.

Theorem 11. If $N$ is a commutative near-ring, then every pseudogroup in $N$ is a semigroup.

Proof. Let $a, b$ be members of some pseudogroup $P(e)P$ in $N$. Then there exist positive integers $n$ and $m$ such that $a^n \in M(e)$ and $b^m \in M(e)$. Hence $(ab)^{nm} = a^{nm}b^{nm} \in M(e)$, so that $P(e)$ is closed with respect to multiplication, and thus is a semigroup.

Theorem 12. Every finite near-ring consists entirely of pseudogroup members.

Proof. Let $N$ be a finite near-ring and let $r$ be the number of elements in $N$. Let $a$ any element in $N$. If $a$ is nilpotent, then $a^n = 0$ for some positive integer $n$, and hence $a$ is in the pseudogroup $P(0)$. Now suppose $a$ is not nilpotent and form the sequence $a, a^2, \ldots, a^r, \ldots$. Since there are only $r$ elements in $N$, including zero, not every element in the sequence is distinct. Let $n$ be the smallest integer such that $a^n = a^{n+m}$ for some integer $m$. We now show that the set $G = \{a^n, a^{n+1}, \ldots, a^{n+m-1}\}$ is a group. We note first that $G$ is a finite semigroup. Further, the cancellation laws hold in $G$, for if $a^{n+r}a^{n+s} = a^{n+t}a^{n+t}$, then $(n+r)+(n+s) = (n+r)+(n+t)(\text{modulom})$. It follows that $n + s = n + t(\text{modulom})$ so that $a^{n+s} = a^{n+t}$. The right cancellation may be shown similarly. But every finite semigroup in which the cancellation laws hold is a group. Hence $G$ is a multiplicative group, so that $a^n$ is a group member in $N$, and $a$ is thus a pseudogroup in $N$.

References

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