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FUZZY NEAR-RING MODULES OVER FUZZY NEAR-RINGS

YOUNG-IN KWON

The concept of a fuzzy subset of a set was first introduced by Zadeh([5]) and several authors including Zadeh have discussed various aspects of the theory and applications of fuzzy sets. In 1982, Liu and Nanda([1],[2]) applied this concept to the theory of rings and ideals. Moreover, in [3], Nanda introduced the concept of a fuzzy module over a fuzzy ring. Now we introduce the notion of a fuzzy near-ring module over a fuzzy near-ring. The proof of the theorems in this papaer is similar to the one in [3].

We first recall some basic definitions for the sake of completeness.

A fuzzy set in a set S is a function A from S into [0,1]. Let A and B be fuzzy sets in a set S. Then we define

$$A = B \iff A(x) = B(x) \qquad \text{for all } x \in S.$$

$$A \subseteq B \iff A(x) \leq B(x) \qquad \text{for all } x \in S.$$

$$(A \cup B)(x) = \max\{A(x), B(x)\} \qquad \text{for all } x \in S.$$

$$(A \cap B)(x) = \min\{A(x), B(x)\} \qquad \text{for all } x \in S.$$

More generally, for a family of fuzzy sets, $\{A_i | i \in I\}$, we define

$$(\cup A_i)(x) = \sup_{i \in I}\{A_i(x)\}, \quad x \in S.$$

$$(\cap A_i)(x) = \inf_{i \in I}\{A_i(x)\}, \quad x \in S.$$

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DEFINITION 1. ([4]) Let θ be a function from a set S into a set T and let A be any fuzzy set in S. The image of A under θ, $\theta(A)$, is the fuzzy set in T defined by

$$[\theta(A)](y) = \begin{cases} \sup_{x\in\theta^{-1}(y)} A(x) & \text{if } \theta^{-1}(y) \text{ is not empty} \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in T$. Let B be any fuzzy set in T. The inverse image of B under θ,$\theta^{-1}(B)$, is the fuzzy set in S defined by

$$[\theta^{-1}(B)](x) = B(\theta(x)) \qquad \text{for all } x \in S.$$

DEFINITION 2. Let R be a near-ring and N a fuzzy set in R. Then N is called a fuzzy near-ring in R if

(1) $N(x+y) \geq \min\{N(x), N(y)\}$,
(2) $N(-x) \geq N(x)$,
(3) $N(xy) \geq \min\{N(x), N(y)\}$, for all $x, y \in R$.

DEFINITION 3. Let R be a near-ring and N a fuzzy near-ring in R. Let Y be a near-ring module over R and M a fuzzy set in Y. Then M is called a fuzzy near-ring module in Y if

(4) $M(x+y) \geq \min\{M(x), M(y)\}$,
(5) $M(\lambda x) \geq \min\{N(\lambda), M(x)\}$, for all $x, y \in Y$ and all $\lambda \in R$.
(6) $M(0) = 1$.

If N is an ordinary near-ring, then condition (5) is replaced by

(7) $M(\lambda x) \geq M(x)$ for all $\lambda \in N$ and all $x \in Y$.

THEOREM 4. *Let Y be a near-ring module over a fuzzy near-ring N in R. Then M is a fuzzy near-ring module in Y if and only if $M(\lambda x+\mu y) \geq \min\{\min\{N(\lambda), M(x)\}, \min\{N(\mu), M(y)\}\}$ for all $\lambda, \mu \in N$ and all $x, y \in Y$.*

If N is an ordinary near-ring,then the above condition is replaced by

$$M(\lambda x + \mu y) \geq \min\{M(x), N(y)\} \qquad \textit{for all } x, y \in Y.$$

THEOREM 5. *Let Y be a near-ring module over an near-ring R with identity. If M is a fuzzy near-ring module in Y and if $\lambda \in R$ is invertible, then $M(\lambda x) = M(x)$ for all $x \in Y$.*

Proof. If $\lambda \in R$ is invertible, then we have for all $x \in Y$,

$$M(x) = M(\lambda^{-1}\lambda x) \geq M(\lambda x) \geq M(x)$$

and so $M(\lambda x) = M(x)$. This completes the proof.

THEOREM 6. *Let $\{M_i | i \in I\}$ be a family of fuzzy near-ring modules in Y. Then $\cap_{i\in I} M_i$ is a fuzzy near-ring modules in Y.*

Proof. Let $M = \cap_{i\in I} M_i$. Then we have for all $\lambda \in R$ and for all $x, y \in Y$,

$$\begin{aligned} M(x+y) &= \inf_{i\in I} M_i(x+y) \\ &\geq \inf_{i\in I}\{\min\{M_i(x), M_i(y)\}\} \\ &= \min\{\inf_{i\in I} M_i(x), \inf_{i\in I} M_i(y)\} \\ &= \min\{M(x), M(y)\}, \end{aligned}$$

and

$$\begin{aligned} M(\lambda x) &= \inf_{i\in I} M_i(\lambda x) \\ &\geq \inf_{i\in I}\{\min\{N(\lambda), M_i(x)\}\} \\ &= \min\{N(\lambda), \inf_{i\in I} M_i(x)\} \\ &= \min\{N(\lambda), M(x)\}. \end{aligned}$$

This completes the proof.

THEOREM 7. *Let Y and W be near-ring modules over a fuzzy near-ring N in a near-ring R and θ a homomorphism of Y into W. Let M* be a fuzzy near-ring module in W. Then the inverse image $\theta^{-1}(M)$ of *M is a fuzzy near-ring module in Y.*

Proof. For all $x, y \in Y$, and all $\lambda, \mu \in R$, we have

$$\begin{aligned}\theta^{-1}(M)(\lambda x + \mu y) &= M(\theta(\lambda x + \mu y)) \\ &= M(\lambda\theta(x) + \mu\theta(y)) \\ &\geq \min\{\min\{N(\lambda), M(\theta(x))\}, \\ &\qquad \min\{\min N(\mu), M(\theta(y))\}\} \\ &= \min\{\min\{N(\lambda), \theta^{-1}(M)(x)\}, \\ &\qquad \min\{N(\mu), \theta^{-1}(M)(y)\}\}.\end{aligned}$$

By Theorem 4, $\theta^{-1}(M)$ is a fuzzy near-ring module in Y. This completes the proof.

We say that a fuzzy set A in M has the sup property if, for any subset T of M, there exists $t_o \in T$ such that $A(t_o) = \sup_{t \in T} A(t)$.

THEOREM 8. *Let Y and W be near-ring modules over a fuzzy near-ring N in a near-ring R and θ a homomorphism of Y into W. Let W be a fuzzy near-ring module in Y that has the sup property. Then the image $\theta(M)$ of M is a fuzzy near-ring module in W.*

Proof. Let $u, v \in W$. If either $\theta^{-1}(u)$ or $\theta^{-1}(v)$ is empty, then the result holds. Suppose that neither $\theta^{-1}(u)$ nor $\theta^{-1}(v)$ is empty. Then we have

$$\begin{aligned}\theta(M)(\lambda u + \mu v) &= \sup_{\omega \in \theta^{-1}(\lambda u + \mu v)} M(\omega) \\ &\geq \min\{\min\{N(\lambda), \theta(M)(u)\}, \min\{N(\mu), \theta(M)(v)\}\}.\end{aligned}$$

This completes the proof.

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Department of Mathematics Education
Gyeongsang National University
Chinju 660-701,Korea