Pusan Kyŏngnam Math. J. 10(1994), No. 1, pp. 59-65

SOME RESULTS OF THE DOMINION IN A WELL-POINTED COMPLETE CATEGORY

SUK BONG YOON

1. Introduction

Recalling the principal definitions in [4], we say that a subsemigroup A of a semigroup B dominates an element d in B if, for any an arbitrary semigroup C and arbitary homomorphisms $B \underset{g}{\overset{f}{\rightrightarrows}} C$, $f(a) = g(a)$ for every a in A implies $f(d) = g(d)$.

The set of elements of B domininated by A is a semigroup of B containing A, which we call the dominion of A.

If the dominion of A is the whole of B we say that A is epimorphically embedded in B (for the inclusion mapping is an epimorphism in the usual categorical sense of being right cancellable).

In [5] it is to extend to categories the basic theorem of [3],[4] describing dominions in (commutative) semigroups and in [6] is a first approximation to a description of epimorphic extensions in the category of finite–dimensional algebras over a field.

In this paper, we shall describe dominions in well–pointed complete category. By the categorical language, we will give slightly different definition of dominion in a well–pointed complete category. Also we will show the some results of dominion by the given definition in Section 3.

2. Preliminaries

In this section, the basic definitions and properties of paticular momomorphisms are recalled ([1],[2],[5]). In [5] Isbell proved the following remarkable theorem.

Received March 14, 1994

ZIGZAG THEOREM. *Let $\mathcal{A}$ be a subcategory of a small category $\mathcal{B}$. Then a morphism $b \in \mathcal{B}$ is in the dominion of $\mathcal{A}$ in $\mathcal{B}$ if and only if for some $m \geq 1$ two factorizations $b = a_0 y_1 = x_m a_{2m}$ connected by relations*

$$a_0 = x_1 a_1, \qquad a_{2m-1} y_m = a_{2m}$$

$$a_{2i-1} y_i = a_{2i} y_{i+1}, \ x_i a_{2i} = x_{i+1} a_{2i+1} \quad (i = 1, 2, \cdots, m-1)$$

with $a_0, a_1, \cdots, a_{2m}$ in $\mathcal{A}$ and $x_1, \cdots, x_m, y_1, \cdots, y_m$ in $\mathcal{B}$.

Such system of equalities is called a zigzag of length m in $\mathcal{B}$ over $\mathcal{A}$ with valued b.

DEFINITION 2.1 ([1],[2]). A morphism $E \xrightarrow{e} A$ in a finitely complete category is called a **regular monomorphism** provided that it is an equalizer of some pair of morphisms.

NOTES 2.2.

(1) It is clear from the uniqueness requirement in the definition of equalizer that regular monomorphisms must be monomorphisms.
(2) One should be aware of the difference between the concepts of equalizer and regular monomorphism. "Equalizer" is defined relative to pairs of morphisms, whereas "regular monomorphism" is absolute. The difference is more that just a technical one. For example, it is possible for a functor to preserve regular monomorphisms without preserving equalizers [1].

EXAMPLES 2.3.

(1) In **Set** the regular monomorphisms are the injective functions, *i.e.* , up to isomorphism, precisely the inclusions of subsets.
(2) In many "algebraic" categories, e.g., in **Gp** and **Vec** all monomorphisms are regular [7]. However, in **Mon**, **Sgr** and **Rng** monomorphisms need not be regular, e.g., the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a non-regular monomorphism.

DEFINITION 2.4 ([1],[2]). A monomorphism m in any small category is called **extremal** provided that it satisfies the following extremal condition: If $m = f \circ e$, where e is an epimorphism, then e must be an isomorphism.

EXAMPLES 2.5.

(1) In the categories **Set**, **Gp**, and **Vec** extremal monomorphisms are the same as monomorphisms.

(2) In **Sgr** and **Rng** there are monomorphisms that are not extremal ([1],[2]).

DEFINITION 2.6 ([1],[2]). Let $\mathcal{M}$ be a class of monomorphisms in any finitely complete category. An $\mathcal{M}$–**subobject** of an object B is a pair (A, m), where $A \xrightarrow{m} B$ belong to $\mathcal{M}$. In case $\mathcal{M}$ consists of all (regular,extremal) monomorphisms, $\mathcal{M}$–subobjects are called (regular, extremal) subobjects.

By the above definition, a subobject of a set B in **Set** is a pair (A, m), where $A \xrightarrow{m} B$ is an injective function. In order for the notion of subobject to correspond more closely to the notion of subset, two subobjects (A, m) and (A', m') should be considered to be essentially the same if $m[A] = m'[A']$. Furthermore a subobject (A, m) of B in **Set** should be considered to be "smaller than" the subobject (A', m') of B provided that $m[A] \subseteq m'[A']$. The following definitions capture these ideas.

DEFINITION 2.7 ([1],[2]). Let (A, m) and (A', m') be subobjects of B.

(1) (A, m) and (A', m') are called **isomorphic** provided that there exists an isomorphism $A \xrightarrow{h} A'$ with $m = m' \circ h$.

(2) (A, m) is said to be **smaller than** (A', m'), denoted by $(A, m) \leq (A', m')$, provided that there exists some (necessarily unique) morphism $A \xrightarrow{h} A'$ with $m = m' \circ h$.

3. Main theorems

In [5] if $A \xrightarrow{f} B$ is a morphism in a category, then the **dominion** of f is a morphism $D \xrightarrow{u} B$ which equalizes any pair $B \rightrightarrows C$ equalized by f, such that if $D' \xrightarrow{u'} B$ is any other such morphism, then there is a unique morphism $D' \xrightarrow{v} D$ satisfying $v \circ u = u'$.

Through this section we assume that $\mathcal{C}$ is a well-pointed complete category. We have equivalent conditions in the following Lemma.

LEMMA. *If a $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ has a factorization $X \xrightarrow{g} D \xrightarrow{d} Y$ where d is a regular monomorphism, then the following are equivalent:*

(1) *for all morphisms α and β, $\alpha \circ f = \beta \circ f$ implies that $\alpha \circ d = \beta \circ d$,*

(2) (D,d) *is the smallest regular subobject of* Y *through which* f *can be factored,*
(3) D *is the intersection of all regular subobjects of* Y *through which* f *can be factored.*

Proof. (1) implies (2). If (R,r) is any regular subobject of Y through which f can be factored. *i.e.* , $f = r\circ\gamma$ for some $\mathcal{C}$-morphism γ. Then r is an equalizer of some pair of morphisms α,β. Since $\alpha\circ r=\beta\circ r$, we have $\alpha\circ r\circ\gamma=\beta\circ r\circ\gamma$ and so $\alpha\circ f=\beta\circ f$. By (1), we have $\alpha\circ d=\beta\circ d$. By the definition of equalizer there is a unique $\mathcal{C}$-morphism δ such that $d = r\circ\delta$. Hence (D,d) is the smallest regular subobject of Y through which f can be factored.

(2) implies (1). Suppose that for any morphisms α and β we have $\alpha\circ f=\beta\circ f$. Let (Q,q) be an equalizer of a pair of given morphisms α, β through which f can be factored by hypothesis. Since $\alpha\circ q=\beta\circ q$, we have $\alpha\circ f=\beta\circ f$. By (2), there exists a $\mathcal{C}$-morphism ϵ such that $d=q\circ\epsilon$. Since $\alpha\circ q\circ\epsilon=\beta\circ q\circ\epsilon$, we have $\alpha\circ d=\beta\circ d$.

Now (2) implies (3). Let (K,k) be any regular subobject of Y through which f can be factored. We have $f=d\circ g=k\circ\delta$ for some $\mathcal{C}$-morphism δ. By (2), there exists a $\mathcal{C}$-morphism γ such that $d=k\circ\gamma$. So we have $k\circ\gamma\circ g = d\circ g = f = k\circ\delta$. Since k and d are monomorphisms, such morphism g is unique such that $\gamma\circ g=\delta$. Hence (D,d) is the intersection of all regular monomorphisms with codomain Y through which f can be factored.

(3) implies (2). By the definition of intersection, (D,d) is the smallest regular subobject of Y through which f can be factored.

We will define slightly different definition of dominion in a well–pointed complete category

DEFINITION. If a $\mathcal{C}$-morphism $X\xrightarrow{f}Y$ has a factorization $X\xrightarrow{g} D\xrightarrow{d}Y$, where d is a regular monomorphism that is characterized uniquely by any of the equivalent conditions in the above Lemma, then (D,d) is called the **dominion** of f, denoted by $Dom(f)$, and $f=d\circ g$ is called the **dominion factorization** of f.

By the new definition of dominion of any morphism we obtain trivial results as the following:

REMARKS.

(1) *Any two dominions of* $X \xrightarrow{f} Y$ *are isomorphic subobjects of* Y.

(2) $X \xrightarrow{f} Y$ *is epimorphism if and only if* (Y, id_Y) *is a dominion of* f.

We can be say the fact that the dominion of any morphism is the (Epi, RegMono)-factorization in the following theorem:

THEOREM 3.1. *The extremal monomorphisms are precisely the regular monomorphisms if and only if the (Epi, ExtrMono)-factorization of any morphism is the same as its dominion factorization.*

Proof. Suppose that a $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ is the (Epi, ExtrMono)-factorization of f. *i.e.* , $f = d \circ \gamma$ for some γ epimorphism and d extremal monomorphism. By the hypothesis, d is a regular monomorphism. If $\alpha \circ f = \beta \circ f$ for any $\mathcal{C}$-morphisms α and β, then we have $\alpha \circ d \circ \gamma = \beta \circ d \circ \gamma$. Since γ is epimorphism, we have $\alpha \circ d = \beta \circ d$. Hence $f = d \circ \gamma$ is the dominion factorization of f.

Now assume that $f = d \circ g$ is the dominion factorization of f. By the hypothesis, d is a extremal monomorphism. It is enough to show that g is an epimorphism. If $\alpha \circ f = \beta \circ f$ for any $\mathcal{C}$-morphisms α and β, we obtain $\alpha \circ d \circ g = \beta \circ d \circ g$. But by the definition of dominion of f, we must have $\alpha \circ d = \beta \circ d$. Since δ is a right cancellable with respect to morphism composition, it is an epimorphism.

Conversely, since the (Epi, ExtrMono)-factorization of any morphism is the same as its dominion factorization, extremal monomorphism and regular monomorphism are coincide.

The following theorem shows that for any regular monomorphism we have its dominion is itself.

THEOREM 3.2. *A* $\mathcal{C}$*-morphism* $X \xrightarrow{f} Y$ *is a regular monomorphism if and only if* (X, f) *is a dominion of* f.

Proof. Let (D, d) be the dominion of f. *i.e.* , $f = d \circ \gamma$ for some $\mathcal{C}$-morphism γ. Since f is a regular monomorphism, (X, f) is a regular subobject of Y through which f can be factored. But since (D, d) is the dominion of f, we have $d = f \circ \delta$ for some $\mathcal{C}$-morphism δ. It is enough to show that γ is an isomorphism. We have $f \circ id_X = f = d \circ \gamma = f \circ \delta \circ \gamma$.

Since f is monomorphism, we have $id_X = \delta \circ \gamma$. Hence γ is a section and so it is an isomorphism.

Conversely, by the definition of dominion of f, it is a regular monomorphism.

The following theorem shows that if any morphism factors as an epimorphism followed by a monomorphism, its dominion is that of the monomorphism.

THEOREM 3.3. *If $f = \delta \circ \gamma$ where γ ia an epimorphism, then the dominion of f and δ coincide.*

Proof. Let (D, d) and (Q, q) are the dominion of f and δ respectively. Since (Q, q) is the dominion of δ, q is an equalizer of the some pair of $\mathcal{C}$-morphisms α, β and $\delta = q \circ \lambda$ for some $\mathcal{C}$-morphism λ. So we have $\alpha \circ q \circ \lambda = \beta \circ q \circ \lambda$. Since $\delta = q \circ \lambda$, $f = \delta \circ \gamma$ and γ an epimorphism, we have $\alpha \circ f = \beta \circ f$. Since (D, d) is the dominion of f, we have $\alpha \circ d = \beta \circ d$. But q is an equalizer of α and β, there exists unique $\mathcal{C}$-morphism ζ such that $d = q \circ \zeta$.

It is enough to show that ζ is an isomorphism. Since (D, d) is the dominion of f, there is a pair $\mathcal{C}$-morphisms μ, ν such that d is an its equalizer and $f = d \circ g$ for some $\mathcal{C}$-morphism g. By the hypothesis, we have $\mu \circ \delta = \nu \circ \delta$. Since (Q, q) is the dominion of δ, we have $\mu \circ q = \nu \circ q$. But d is an equlaizer of μ and ν, there exists a unique $\mathcal{C}$-morphism ξ such that $q = d \circ \xi$. So we have $d = q \circ \zeta$ and $q = d \circ \xi$. Since ζ is a section and a retraction, it is an isomorphism.

The following theorem shows that the dominion of any morphism is the same as the equalizer of its cokernel pair.

THEOREM 3.4. *Consider the pushout square in $\mathcal{C}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mu} \\ Y & \xrightarrow[\nu]{} & Q \end{array}$$

Then (D, d) is the dominion of f if and only if d is an equalizer of μ and ν.

Proof. Since d is a regular monomorphism, there is a pair $\mathcal{C}$-morphisms α and β such that d is an equalizer of α and β and $f = d \circ g$ for some $\mathcal{C}$-morphism g. Since $\alpha \circ d = \beta \circ d$, we have $\alpha \circ f = \beta \circ f$. By the definition of pushout, there is a unique $\mathcal{C}$-morphism ξ such that $\alpha = \xi \circ \mu$ and $\beta = \xi \circ \nu$. Since $\mu \circ f = \nu \circ f$ and d is the dominion of f, we have $\mu \circ d = \nu \circ d$. If γ is a $\mathcal{C}$-morphism with $\mu \circ \gamma = \nu \circ \gamma$, we have $\alpha \circ \gamma = \beta \circ \gamma$. But d is an equalizer of α and β, there exists a unique $\mathcal{C}$-morphism λ such that $\gamma = d \circ \lambda$. Hence d is an equalizer of μ and ν.

Conversely, by the hypothesis, we have $\mu \circ d = \nu \circ d$. If $\phi \circ f = \psi \circ f$ for any $\mathcal{C}$-morphisms ϕ and ψ, by the definition of pushout, there exists a unique $\mathcal{C}$-morphism χ such that $\phi = \chi \circ \mu$ and $\psi = \chi \circ \nu$. So we have $\phi \circ d = \psi \circ d$. Hence (D, d) is the dominion of f.

NOTE. We can be define the dual concepts –codominion (factorization) – of dominion of any morphism and describe its dual theorems.

References

1. J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and concrete categories*, John Wiley and Sons, Inc., New york, 1990
2. H. Herrlich and G.E. Strecker, *Category theory*, Allyn and Bacon Inc., Boston, 1973.
3. J.R. Howie and J.R. Isbell, *Epimorphisms and dominions. II*, J. Algebra 6 (1967), 7–21.
4. J.R. Isbell, *Epimorphisms and dominions*, Proc. Conf. Categorical Algebra (La Jolla, Calif.,1965), Springer, New York (1966), 232–246
5. ———, *Epimorphisms and dominions. III*, Amer J. Math. 90 (1968), 1025–1030.
6. ———, *Epimorphisms and dominions IV*, J. London Math. Soc. (2) 1 (1969), 265–273.
7. C.E. Linderholm, *A group epimorphism is surjective*, Amer. Math Monthly 77 (1970), 176–177.

Department of Mathematics
Dongeui University
Pusan 614-714, Korea