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MONOTONICITY OF HYPERBOLIC CURVATURE

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1. Introduction

Let Ω be a hyperbolic region in the complex plane $\mathbf{C}$ and $K_\Omega(a,\gamma)$ denote the hyperbolic curvature of a C^2 curve γ in Ω at a point $a \in \gamma$. Flinn and Osgood [3] established a monotonicity property for the hyperbolic curvature. They proved that if Ω is a simply connected subregion of a simply connected hyperbolic region Δ, then for any C^2 curve γ in Ω

$$\max\{K_\Omega(a,\gamma),2\} \leq \max\{K_\Delta(a,\gamma),2\}.$$

They also showed that the monotonicity property would not extend to arbitrary hyperbolic regions.

In this paper we show that the conclusion of the Flinn-Osgood Monotonicity Theorem remains valid for arbitrary hyperbolic regions provided that the group homomorphism $g_* : \pi(\Omega,a) \to \pi(\Delta,a)$ induced by the inclusion mapping $g : \Omega \to \Delta$ is a monomorphism.

2. Universal covering projections

Let D be the open unit disk in the complex plane $\mathbf{C}$. Suppose Ω is a hyperbolic region $\mathbf{C}$ and $a \in \Omega$. Then there exists a holomorphic universal covering projection $f : (D,0) \to (\Omega,a)$. This is called the General Riemann Mapping Theorem (see [1, p.142] or [2, p.39]); in case Ω is simply connected this is the Riemann Mapping Theorem. We shall need the following properties of a covering projection (see [4, Ch. 5] or [8, Ch. 3]).

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(1) Given any path γ in Ω with initial point a, there is a unique path $\widetilde{\gamma}$ in D with initial point 0 such that $f \circ \widetilde{\gamma} = \gamma$. The path $\widetilde{\gamma}$ is called the lift of γ via f.

(2) Suppose γ_1, γ_2 are two paths in Ω from a to the common terminal point b. Let $\widetilde{\gamma}_j$ be the unique lift of γ_j via f with initial point 0. Then $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the same terminal point if and only if γ_1 is homotopic to γ_2 in Ω with fixed end points.

(3) Suppose γ is a closed path in Ω based at a and $\widetilde{\gamma}$ is the lift of γ via f with initial point 0. Then $\widetilde{\gamma}$ is a closed path if and only if γ is null homotopic.

(4) If $g : (D, 0) \to (\Omega, a)$ is any holomorphic function, then there is a unique holomorphic function $\widetilde{g} : (D, 0) \to (D, 0)$ such that $f \circ \widetilde{g} = g$. The function $\widetilde{g}$ is called the lift of g relative to f.

We briefly indicate the construction of $\widetilde{g}$. For $\widetilde{z} \in D$ let $\widetilde{\gamma}$ be any path in D from 0 to $\widetilde{z}$. Then $\gamma = g \circ \widetilde{\gamma}$ is a path in Ω from a to $z = g(\widetilde{z})$. Since f is a covering projection, there is a unique lift $\widetilde{\delta}$ of γ in D via f with initial point 0. Let $\widetilde{w}$ be the terminal point of $\widetilde{\delta}$. Then define $\widetilde{g}(\widetilde{z}) = \widetilde{w}$. It remains to show that $\widetilde{g}$ is well-defined. Suppose $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are both paths in D from 0 to $\widetilde{z}$. Then $\gamma_j = g \circ \widetilde{\gamma}_j$ $(j = 1, 2)$ are paths in Ω from a to z. Since D is simply connected, $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$ in D. It follows that γ_1 is homotopic to γ_2 in Ω. Let $\widetilde{\delta}_j$ be the lift of γ_j via f with initial point 0. Then $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ have the same terminal point since γ_1 is homotopic to γ_2. This proves that $\widetilde{g}$ is well-defined.

We shall employ this lifting property in the special case where $f : (D, 0) \to (\Omega, a)$ and $h : (D, 0) \to (\Delta, a)$ are covering projections, $\Omega \subset \Delta$ and $g : \Omega \to \Delta$ is the inclusion map. Then $g \circ f : (D, 0) \to (\Delta, a)$ has a lift $\widetilde{g}$ via h.

The fundamental group of Ω with base point a will be denoted by $\pi(\Omega, a)$. For a closed path γ based at a, $[\gamma]$ is the homotopy class determined by γ. A continuous function $g : (\Omega, a) \to (\Delta, b)$ induces a group homomorphism $g_* : \pi(\Omega, a) \to \pi(\Delta, b)$ defined by $g_*([\gamma]) = [g \circ \gamma]$.

The following result is well known (see [6]). We include a proof for the convenience of the reader.

Theorem 1. *Suppose Ω and Δ are hyperbolic regions in $\mathbf{C}$ with $\Omega \subset \Delta$ and $a \in \Omega$. Let $g : \Omega \to \Delta$ be the inclusion map and $g_* : \pi(\Omega, a) \to \pi(\Delta, a)$ the induced group homomorphism. Assume that $f : (D, 0) \to (\Omega, a)$ and $h : (D, 0) \to (\Delta, a)$ are holomorphic universal covering projections. If g_* is a monomorphism, then there exists a conformal mapping $\widetilde{g}$ of $(D, 0)$ into itself such that $f = g \circ f = h \circ \widetilde{g}$.*

Proof. We already know that a holomorphic function $\widetilde{g} : (D, 0) \to (D, 0)$ exists such that $g \circ f = h \circ \widetilde{g}$. All that remains is to show that $\widetilde{g}$ is one-to-one. Suppose $\widetilde{z}_1, \widetilde{z}_2 \in D, \widetilde{z}_1 \neq \widetilde{z}_2$ and $\widetilde{g}(\widetilde{z}_1) = \widetilde{g}(\widetilde{z}_2)$. Let $\widetilde{\gamma}_j$ be the radial path in Ω from 0 to $\widetilde{z}_j$ $(j = 1, 2)$. Then $\gamma_j = f \circ \widetilde{\gamma}_j$ is a path in Ω from a to $f(\widetilde{z}_j)$. Note that

$$f(\widetilde{z}_1) = h(\widetilde{g}(\widetilde{z}_1)) = h(\widetilde{g}(\widetilde{z}_2)) = f(\widetilde{z}_2),$$

so that γ_1, γ_2 both end at the same point. Because $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ do not have the same endpoint but do have the same initial point, the paths γ_1, γ_2 are not homotopic in Ω. Hence, $[\gamma_1 * \gamma_2^{-1}]$ is nontrivial in $\pi(\Omega, a)$. Since g_* is a monomophism, we conclude that $[\gamma_1 * \gamma_2^{-1}]$ is also nontrivial in $\pi(\Delta, a)$, or γ_1 and γ_2 are not homotopic in Δ. If $\widetilde{\delta}_j = \widetilde{g} \circ \widetilde{\gamma}_j$, then

$$h \circ \widetilde{\delta}_j = h \circ \widetilde{g} \circ \widetilde{\gamma}_j = f \circ \widetilde{\gamma}_j = \gamma_j.$$

Thus, $\widetilde{\delta}_j$ is a lift of γ_j via the covering $h : (D, 0) \to (\Delta, a)$ and 0 is the initial point of $\widetilde{\delta}_j$. Because γ_1 is not homotopic to γ_2 in Δ, it follows that $\widetilde{\delta}_1$and $\widetilde{\delta}_2$ must have distinct endpoints. This contradicts the fact that both $\widetilde{\delta}_1$and $\widetilde{\delta}_2$ end at $\widetilde{g}(\widetilde{z}_1) = \widetilde{g}(\widetilde{z}_2)$. This contradiction shows that $\widetilde{g}$ must be injective.

Remark. For multiply connected regions $\Omega \subset \Delta$ there is a simple geometric criterion for g_* to be a monomorphism. The condition is that every hole in Ω must contain at least one hole of Δ.

3. Hyperbolic curvature

We begin by recalling a few basic facts about the hyperbolic curvature. We refer the reader to [5], [6], and [7] for further details. Let $\lambda_\Omega(z)|dz|$ be the hyperbolic metric on the hyperbolic region Ω. If γ is a C^2 curve in a hyperbolic region Ω with parametrization $z = z(t)$, then the hyperbolic curvature of γ at a point $z = z(t)$ is given by

$$K_\Omega(z,\gamma) = \frac{1}{\lambda_\Omega(z)}\left[K_e(z,\gamma) + 2Im\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right],$$

where

$$K_e(z,\gamma) = \frac{1}{|z'(t)|}Im\left\{\frac{z''(t)}{z'(t)}\right\}$$

denotes the euclidean curvature of γ at $z = z(t)$. Because the hyperbolic metric is invariant under holomorphic covering projections, the same is true of the hyperbolic curvature. That is, $K_\Omega(z,\gamma) = K_\Delta(f(z), f\circ\gamma)$ if Ω and Δ are hyperbolic regions and $f:\Omega\to\Delta$ is a holomorphic covering projection of Ω onto Δ.

Lemma. *Suppose Ω and Δ are hyperbolic simply connected regions in $\mathbf{C}$. If g is a conformal mapping of Ω onto $g(\Omega)\subset\Delta$, then for any path γ in Ω*

$$\max\{K_\Omega(a,\gamma), 2\} \le \max\{K_\Delta(g(a), g\circ\gamma), 2\}.$$

Proof. Since the hyperbolic curvature is a conformal invariant,

$$K_\Omega(a,\gamma) = K_{g(\Omega)}(g(a), g\circ\gamma).$$

The Flinn-Osgood Monotonicity Theorem yields

$$\max\left\{K_{g(\Omega)}(g(a), g\circ\gamma), 2\right\} \le \max\{K_\Delta(g(a), g\circ\gamma), 2\}$$

so this establishes the lemma.

We can now state our main result.

Theorem 2. *Suppose Ω and Δ are hyperbolic regions, $\Omega \subset \Delta$ and $a \in \Omega$. If $g : \Omega \to \Delta$ is the inclusion map and $g_* : \pi(\Omega, a) \to \pi(\Delta, a)$ is a monomorphism, then for any path γ through a,*

$$\max\{K_\Omega(a,\gamma), 2\} \le \max\{K_\Delta(a,\gamma), 2\}.$$

Proof. We need only consider the case in which $K_\Omega(a,\gamma) \ge 2$. Let $f : (D, 0) \to (\Omega, a)$ and $h : (D, 0) \to (\Delta, a)$ be holomorphic universal covering projections. Since g_* is a monomorphism, it follows from Theorem 1 that there is a conformal mapping $\widetilde{g}$ of $(D, 0)$ into itself such that $g \circ f = h \circ \widetilde{g}$. Let $\widetilde{\gamma}$ be the lift of γ via f with initial point 0. Then $\widetilde{\delta} = \widetilde{g} \circ \widetilde{\gamma}$ is the lift of γ via h with initial point 0. The invariance of hyperbolic curvature under holomorphic coverings implies that

$$K_\Omega(a,\gamma) = K_D(0,\widetilde{\gamma}), \; K_\Delta(a,\gamma) = K_D\left(0,\widetilde{\delta}\right),$$

so it suffices to show that

$$K_D(0,\widetilde{\gamma}) \le K_D\left(0,\widetilde{\delta}\right).$$

Since $\widetilde{g}$ is a conformal mapping of $(D, 0)$ into itself and $\widetilde{g} \circ \widetilde{\gamma} = \widetilde{\delta}$, $K_D(0,\widetilde{\gamma}) \ge 2$, this is a consequence of previous Lemma.

If Ω is a simply connected subregion of a hyperbolic region Δ, then $\pi(\Omega, a) = 1$ for each $a \in \Omega$. Hence the induced group homomorphism $g_* : \pi(\Omega, a) \to \pi(\Delta, a)$ is a monomorphism. Thus, we obtain the following result.

Corollary. *Suppose Δ is a hyperbolic region in $\mathbf{C}$ and Ω is a simply connected subregion of Δ. If γ is a path in Ω, then for all $z \in \gamma$*

$$\max\{K_\Omega(z,\gamma), 2\} \le \max\{K_\Delta(z,\gamma), 2\}.$$

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