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# ON THE FIVE–FOLD FACTORIZATIONS

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## 1. Introduction

Every function$X \xrightarrow{f} Y$ can be factored through its image, *i.e.*, written as a composite $X \xrightarrow{f} Y = X \xrightarrow{e} f[X] \xrightarrow{m} Y$, where $X \xrightarrow{e} f[X]$ is the codomain-restriction of $f$ and $f[X] \xrightarrow{m} Y$ is the inclusion. This fact, through simple, is often useful. Similarly, in constructs such as **Vec**, **Gp** and **Top** every morphism can be factored through its "image". Since

(1) for categories in general, no satisfactory concept of "embedding of subobjects" and hence of "image of a morphism" is available,

and

(2) for certain constructs, factorizations of morphisms different from the one through the image are of interest (e.g., in **Top** the one through the closure of the image: $X \xrightarrow{f} Y = X \to cl_Y f[X] \hookrightarrow Y$),

categorists have created an axiomatic theory of factorization structures $(\mathcal{E},\mathcal{M})$ for morphisms of a category $\mathcal{C}$ where $\mathcal{E}$ and $\mathcal{M}$ are classes of $\mathcal{C}$-morphisms such that each $\mathcal{C}$-morphism has an $(\mathcal{E},\mathcal{M})$-factorization $X \xrightarrow{f} Y = X \xrightarrow{e\in\mathcal{E}} Z \xrightarrow{m\in\mathcal{M}} Y$ ([1]-[9]). Naturally, without further assumptions on $\mathcal{E}$ and $\mathcal{M}$ such factorizations might be quite useless. A careful analysis has revealed that the crucial requirement that cases $(\mathcal{E},\mathcal{M})$- factorizations to have appropriate characteristics is the so-called "unique $(\mathcal{E},\mathcal{M})$-diagonalization" condition, described in definition of section 2. Such factorization structures for morphisms have turned out to be useful, especially for "well-behaved" categories (e.g., those having products and satisfying suitable smallness conditions).

The aim of this paper is to give a unique five-fold factorization of any morphism in a well-powered, co-(well-powered), complete, cocomplete category.

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## 2. Preliminaries

In this section factorization structures for morphisms are defined and investigated ([1]-[10]).

DEFINITION ([1]-[9]). Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. $(\mathcal{E},\mathcal{M})$ is called a factorization structure for morphisms in $\mathcal{C}$ and $\mathcal{C}$ is called $(\mathcal{E},\mathcal{M})$-structured provided that

(1) each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms,
(2) $\mathcal{C}$ has $(\mathcal{E},\mathcal{M})$-factorizations (of morphisms); *i.e.*, each morphism $f$ in $\mathcal{C}$ has a factorization $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,
(3) $\mathcal{C}$ has the unique $(\mathcal{E},\mathcal{M})$-diagonalization property; *i.e.*, for each commutative square

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[m]{} & W \end{array} \qquad \text{with } e \in \mathcal{E} \text{ and } m \in \mathcal{M},$$

there exists a unique diagonal, *i.e.*, a morphism $d$ such that $d \circ e = f$ and $m \circ d = g$.

In [2], we can be explain the examples of factorization structure for any morphism.

We conclude our $(\mathcal{E},\mathcal{M})$-structured criteria as the following theorem.

THEOREM ([1]-[3]). *If $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{C}$, then $\mathcal{C}$ is $(\mathcal{E},\mathcal{M})$-structured if and only if the following conditions are satisfied*

(1) *Iso($\mathcal{C}$)$\subseteq \mathcal{E} \cap \mathcal{M}$,*
(2) *each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition,*
(3) *$\mathcal{C}$ has the $(\mathcal{E},\mathcal{M})$-factorization property, unique in the sense that for any pair of $(\mathcal{E},\mathcal{M})$-factorizations $m_1 \circ e_1 = f = m_2 \circ e_2$ of a morphism $f$ there exists a unique isomorphism $h$ such that $h \circ e_1 = e_2$ and $m_2 \circ h = m_1$.*

The following theorem shows that if $\mathcal{C}$ is a well-powered that has intersections, equalizers and pullbacks, then it is an (ExtrEpi, Mono)-structured.

THEOREM ([1],[5]). *If $C$ is well-powered, finitely complete category and has intersections, then $C$ is an (ExtrEpi, Mono)-structured.*

The following theorem shows that if $C$ is well-powered that has intersections and equalizers, then it is an (Epi, ExtrMono)-structured

THEOREM ([1],[5]). *If $C$ is well-powered category that has intersections and equalizers, then $C$ is an (Epi, ExtrMono)-structured.*

For the existence of the factorization we have the following theorem.

THEOREM ([1],[5]). *If $C$ is a well-powered, finitely complete category that has intersections, then each $C$-morphism $f$ has a factorization (which is unique up to isomorphisms) of the form $f = m \circ b \circ e$, where $e$ is an extremal epimorphism, $b$ is a bimorphism and $m$ is an extremal monomorphism.*

The following theorem shows that if any morphism factors as an epimorphism followed by a monomorphism, its dominion is that of the monomorphism in a well-powered complete category.

THEOREM ([10]). *If $f = \delta \circ \gamma$ where $\gamma$ is an epimorphism, then the dominion of $f$ and $\delta$ coincide.*

We can be obtains the dual result of the above theorem as the following:
*Let $C$ be a well-powered, co-(well-powered) and bicomplete category and if $f = \delta \circ \gamma$ where $\delta$ is a monomorphism, then the codominion of $f$ and $\gamma$ coincide.*

## 3. Main theorems

Throughout this section, $C$ denotes a well-powered, co-(well-powered), complete, cocomplete category.

Using the dominion and the codominion of any $C$-morphism we have the five-fold factorization of any $C$-morphism as the following theorem.

THEOREM 3.1. *A $C$-morphism $X \xrightarrow{f} Y$ has a unique five-fold factorization: $f = \alpha \circ \beta \circ \delta \circ \xi \circ \eta$ where $(\eta, cod(\eta))$ is the codominion of $f$, $\xi$ is* *an extremal epimorphism, $\delta$ is a bimorphism, $\beta$ is an extremal monomorphism and $(dom(\alpha), \alpha)$ is the dominion of $f$.*

*Proof.* Since $\mathcal{C}$ is a well-powered complete category, we have any $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ is the uniquely (ExtrEpi, Bi, ExtrMono)-factorization of $f$. i.e., $f = \psi \circ \delta \circ \phi$ for some $\mathcal{C}$-morphisms extremal epimorphism $\psi$, bimorphism $\delta$ and extremal monomorphism $\phi$. Since $f = \psi \circ (\delta \circ \phi)$ and $\delta \circ \phi$ is an epimorphism, we have the dominion of $f$ and $\psi$ are coincide ([10]). If $\alpha$ is the dominion of $\psi$, then we have $\psi = \alpha \circ \beta$ for some $\mathcal{C}$-morphism $\beta$. Since $\psi = \alpha \circ \beta$ is an extremal monomorphism, $\beta$ is an extremal monomorphism ([1],[5],[10]).

Since $\mathcal{C}$ is a well-powered, co-well-powered, bicomplete category, there is the codominion of any $\mathcal{C}$-morphism. Since $f = (\alpha \circ \beta \circ \delta) \circ \phi$ and $\alpha \circ \beta \circ \delta$ is a monomorphism, we have the fact that the codominion of $f$ is precisely that of $\phi$ ([10]). If $\eta$ is the codominion of $\phi$, then we have $\phi = \xi \circ \eta$ for some $\mathcal{C}$-morphism $\xi$. Since $\phi = \xi \circ \eta$ is an extremal epimorphism, $\xi$ is an extremal epimorphism. Hence $f = \alpha \circ \beta \circ \delta \circ \xi \circ \eta$ is the uniquely (Codom($f$), ExtrEpi, Bi, ExtrMono, Dom($f$))-factorization of $f$.

We can obtain the next corollaries directly from Theorem 3.1.

COROLLARY 3.2. *Any $\mathcal{C}$-morphism $f$ is the unique (Epi, ExtrMono), (ExtrEpi, Mono) and (ExtrEpi, Bi, ExtrMono)- factorizations in terms of the five-fold factorization.*

COROLLARY 3.3. *The five-fold factorization is functorial in a categorical language.*

In order to prove of Theorem 3.5, we need the following Lemma.

LEMMA 3.4. *If $\mathcal{C}$ has the properties that every monomorphism is regular and every epimorphism is regular, then the middle three terms in the five-fold factorization of a $\mathcal{C}$-morphism are isomorphisms.*

*Proof.* Since the dominion of $f$ and $\psi$ are coincide, we have $\psi = \alpha \circ \beta$ where $\alpha$ is a regular monomorphism and $\beta$ is an epimorphism. By the extremal condition of $\psi$, $\beta$ is an isomorphism. Also, since the codominion of $f$ is precisely that of $\phi$, we have $\phi = \xi \circ \eta$ where $\eta$ is a regular epimorphism and $\xi$ is a monomorphism. By the extremal condition of $\phi$, $\xi$ is an isomorphism.

Now, by the five-fold factorization of any $\mathcal{C}$-morphism we have $f = (\alpha\circ\beta\circ\delta)\circ(\xi\circ\eta)$ where $\alpha\circ\beta\circ\delta$ is a monomorphism. By the hypothesis, $(\alpha\circ\beta)\circ\delta$ is an extremal monomorphism. Since $\delta$ is an epimorphism and by the extremal condition of $(\alpha\circ\beta)\circ\delta$, $\delta$ is an isomorphism. Hence the middle three terms of the five-fold factorization of any $\mathcal{C}$-morphism are isomorphisms. *i.e.*, $f = \alpha\circ\beta\circ\delta\circ\xi\circ\eta$ where $\beta$, $\delta$ and $\xi$ are isomorphism.

So the following theorem shows that every $\mathcal{C}$-morphism is unique (up to isomorphism) (Epi, Mono)-factorization if $\mathcal{C}$ has the properties that every monomorphism is regular and every epimorphism is regular.

THEOREM 3.5. *If $\mathcal{C}$ has the properties that every monomorphism is regular and every epimorphism is regular, then every morphism in $\mathcal{C}$ has a natural unique (Epi, Mono)-factorization.*

*Proof.* By the Lemma 3.4, we have the fact that any $\mathcal{C}$-morphism $f$ is unique (up to isomorphism) (RegEpi, RegMono)-factorization of $f$. By the hypothesis, $f$ is unique (up to isomorphism) (Epi, Mono)-factorization of $f$.

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