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DIFFERENTIAL INEQUALITY AND CARATHEODORY FUNCTIONS

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I. Introduction

Let P be the class of functions $p(z)$ which are analytic in the unit disk $D = \{z : |z| < 1\}, p(0) = 1$ and $Rep(z) > 0$ in D. If $p(z)$ is in P, we say $p(z)$ a Caratheodory function.

Miller, Mocanu and Reade [5] and Lewandowski, Miller and Zlotkiewicz [3] and Miller [4] showed that an analytic function satisfying a differential inequality of a certain type is necessarily a Caratheodory function and usefull results.

Lee and Nunokawa [1] showed that an analytic function satisfying a differential inequality a Caratheodory function.

In this paper, we generalize these results. We need the following lemma.

Lemma [1]. *Let $p(z)$ be analytic in D, $p(0) = 1$ and suppose that*

$$Re[p(z) + zp'(z)] > 0 \qquad \text{in } D.$$

Then we have

$$Rep(z) > log\frac{4}{e} \qquad \text{in } D.$$

2. Main Theorems.

Theorem 1 *Let $p(z)$ be analytic in D, $p(0) = 1$ and suppose that*

$$Re[p(z) + zp'(z)] > \beta \qquad \text{in } D,$$

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where $\beta < 1$. *Then we have*

$$Rep(z) > (1-\beta)log\frac{4}{e} + \beta \qquad in\ D.$$

Proof. Let us put

$$q(z) = \frac{p(z)-\beta}{1-\beta}$$

Then $q(z)$ is analytic in D, and $q(0) = 1$. If

$$Re[p(z) + zp'(z)] > \beta$$

then

$$\frac{1}{1-\beta}Re[(p(z)-\beta) + z(p(z)-\beta)'] \\ = Re[q(z) + zq'(z)] > 0$$

From Lemma, we have

$$Req(z) = \frac{Re(p(z)-\beta)}{1-\beta} > log\frac{4}{e}.$$

This shows that

$$Rep(z) > (1-\beta)log\frac{4}{e} + \beta.$$

Remark. Nunokawa [2] proved Theorem 1 by a different method.

From Theorem 1, we get easily Corollary 2;
Corollary 2. *Let* $p(z)$ *be analytic in* D, $p(0) = 1$ *and suppose that*

$$Re[p(z)^2 + 2zp'(z)p(z)] > \beta \qquad in\ D.$$

where $\beta < 1$. *Then we have*

$$Rep(z)^2 > (1-\beta)log\frac{4}{e} \qquad in\ D.$$

Corollary 3. *Let $p(z)$ be analytic in D, $p(0) = 1$, $p(z) \neq 0$ in D and*

$$Re[\frac{1}{p(z)} - \frac{zp'(z)}{p(z)^2}] > \beta \qquad \text{in } D.$$

, where $\beta < 1$. Then we have

$$0 < Rep(z) < ((1-\beta)log\frac{4}{e} + \beta)^{-1} \qquad \text{in } D.$$

Proof. Since

$$Re[\frac{1}{p(z)} - \frac{zp'(z)}{p(z)^2}] = Re[\frac{1}{p(z)} - z(\frac{1}{p(z)})'],$$

from Theorem 1,

$$Re\frac{1}{p(z)} > (1-\beta)log\frac{4}{e} + \beta.$$

Hence we have

$$Re[o < p(z) < ((1-\beta)log\frac{4}{e} + \beta)^{-1} \qquad \text{in } D.$$

Corollary 4. *Let $p(z)$ be analytic in D, $p(0) = 1$, $p(z) \neq 0$ and suppose that*

$$1 + log|p(z)| + Re\frac{zp'(z)}{p(z)} > \beta \qquad \text{in } D.$$

Then we have

$$|p(z)| > (\frac{4}{e^2})^{1-\beta} \qquad \text{in } D.$$

Proof. Since

$$1 + log|p(z)| + Re\frac{zp'(z)}{p}(z)$$
$$= Re[log(ep(z)) + z(log(ep(z)))'],$$

we have

$$\begin{aligned} Relog[ep(z)] &= loge|p(z)| \\ &> (1-\beta)log\frac{4}{e} + \beta \\ &= log(\frac{4}{e})^{1-\beta}e^{\beta}. \end{aligned}$$

This shows that

$$|p(z)| > (\frac{4}{e})^{1-\beta}e^{\beta-1} = (\frac{4}{e^2})^{1-\beta}$$

Remark. Putting $\beta = 0$ in Theorem 1, Corollary 2, 3, 4, we have the results in [1].

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