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REMARKS ON FINITE NORMALIZING EXTENSION RINGS OF GRADED RINGS

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1.Introduction

Generally a finite normalizing extension ring S of a graded ring R is not a graded ring. In this paper we will give some conditions that a finite normalizing extension ring of a graded ring is also a graded ring. An example of this paper shows that this condition is adequate. Under this condition we can find some properties for graded ideals and graded modules. We will prove that a result of L.Souief's paper is also true in graded ring case.

Let G be a multiplicative group with identity element e. A ring R is said to be a graded ring of type G if there is a family of addititve subgroups of R, say $\{R_g \mid g \in G\}$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_k \subset R_{gk}$ for all $g, k \in G$, where we denoted by $R_g R_k$ the set of all finite sums of products $r_g r_k$ with $r_g \in R_g$ and $r_k \in R_k$.

An R-module M is said to be a graded left R-module if there is a family of additive subgroups of M, say $\{M_g \mid g \in G\}$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_k \subset M_{gk}$. It is well known that R_e is a subring and the identity 1 of R is contained in R_e. And also we know that every M_g is a left R_e-module.

For example the group ring $R = A[G]$ is a graded ring where G is a group and R is a ring. The element of $h(R) = \bigcup_{g \in G} R_g$ and $h(M) = \bigcup_{g \in G} M_g$ are called homogeneous elements of R and M respectively. If a nonzero m is contained in M_g, we call m a homogeneous element of degree g and we write $deg(m) = g$. Of course any nonzero element of a graded ring or a graded left R-module has a unique expression as a sum of homogeneous elements.

A submodule N of a module M is a graded submodule if $N = \bigoplus(N \cap M_g)$ or equivalently, if for every $x \in N$ the homogeneous components

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of x are again in N. Similary we define a graded (left) ideal of a ring R.

On the other hand we call an overring S of a ring R a finite normalizing extension of R if there is a finite set $\{x_i\}$ such that $S = \sum_{i=1}^{n} Rx_i$ and $Rx_i = x_iR$ for all i. In this case the set $\{x_i\}$ is called a normalizing base for S.

Generally a finite normalizing extension ring of a graded ring is not a graded ring. But the following theorem shows that there exists a graded ring which is a finite normalizing extension of a graded ring.

THEOREM 1. *Let R be a graded ring of type G and S be a finite normalizing extension ring of R with normalizing base $\{x_i\}$. If for every $1 \leqq i, j \leqq n$, every $g, k \in G$, the product $(R_g x_i)(R_k x_j)$ is contained in $\sum_{\ell=1}^{n} R_{g_k} x_\ell$, and $(\sum R_g x_i) \cap (\sum R_k x_i) = \{0\}$ for $g \neq k$, then S is a graded ring of type G.*

Proof. Let $S_g = \sum R_g x_i$. Then S is the direct sum of the family of subgroups S_g for $S_g \cap S_k = \{0\}$ for $g \neq k$. In fact every s in S is of the form $r_1 x_1 + \cdots + r_n x_n$ for some $r_i \in R$ and each r_i has a unique expression as a sum of homogeneous elements. On the other hand $S_g S_k \subset S_{gk}$ for every $g, k \in G$ since $(R_g x_i)(R_k x_j) \subset \sum R_{gk} x_\ell$.

An example of a finite normalizing extension of a ring satisfying the above conditions is a semigroup ring where the base semigroup is finite. But the following corollary shows that there exist some others which satisfy the above conditions.

COROLLARY 2. *Let R be a graded ring of type G. If S is a finite normalizing extension of R with normalizing base $\{x_i\}$ satisfying*

(1) *for every $g \in G$, $R_g x_i = x_i R_g$ for all i*
(2) *for every i, j, $x_i x_j$ is contained in $\sum R_e x_i$*
(3) *for every $g, k \in G$, $\sum R_g x_i \cap \sum R_k x_i = \{0\}$*

Proof. Since $(R_g x_i)(R_k x_j) \subset R_g R_k x_i x_j \subset R_{gk} \sum R_e x_\ell \subset \sum R_{gk} x_\ell$, the conditions of corollary 2 satisfies the conditions of theorem 1.

An example of a finite normalizing extension satisfying the condition of corollary 2 is a free liberal extension satisfying (2). The following example shows that there are some other finite normalizing extension rings satisfying the conditions of corollary 2 which are not semigroup ring neither a free liberal extension.

EXAMPLE 3. Let A be a graded ring of type G. Let R be the ring of two by two diagonal matrices over A that is

$$R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ and } \qquad x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad x_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If we define S as $S = Rx_1 + Rx_2 + Rx_3 + Rx_4$ where the operation is matrix multiplication, then S is the ring of two by two matrices over A and a finite normalizing extension of R. Clearly R and S are graded rings of type G where

$$R_g = \begin{pmatrix} A_g & 0 \\ 0 & A_g \end{pmatrix} \text{ and } \qquad S_g = \begin{pmatrix} A_g & A_g \\ A_g & A_g \end{pmatrix}$$

But S is not liberal extension neither free. In fact

$$\begin{pmatrix} 0 & r_g^1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r_g^1 & 0 \\ 0 & r_g^2 \end{pmatrix} x_2 \neq x_2 \begin{pmatrix} r_g^1 & 0 \\ 0 & r_g^2 \end{pmatrix} = \begin{pmatrix} 0 & r_g^2 \\ 0 & 0 \end{pmatrix}$$

$$for \begin{pmatrix} r_g^1 & 0 \\ 0 & r_g^2 \end{pmatrix} \in R_g$$

and $\begin{pmatrix} 0 & 0 \\ 0 & r_2 \end{pmatrix} x_1 + \begin{pmatrix} r_1 & 0 \\ 0 & -r_2 \end{pmatrix} x_4 = 0 \ for\ every\ r_1, r_2 \in A.$

But S satisfies the conditions of corollary 2. In fact $(R_g x_2)(R_k x_3)$ and $(R_g x_3)(R_k x_2)$ are contained in $R_{gk} x_1$ and $S_g \cap S_k = 0$ for $g \neq k$.

In this paper we assume that R is a graded ring of type G and S is a finite normalizing extension of R satisfying the conitions of corollary 2. At first we get the following propositions.

PROPOSITION 4.

(1) *If I is a graded ideal of S, then $I \cap R$ is a graded ideal of R.*
(2) *If I is a graded ideal of R, then SI is a graded left ideal of S and SIS is a graded ideal of S.*

Proof. (1)For every $a \in I \cap R$, a has a unique expression as a sum of homogeneous components that is $a = a_{g_1} + a_{g_2} + \cdots + a_{g_t}$ where $a_{g_i} \in S_{g_i}$. Since a is contained in R each a_{g_i} is also contained in R and contained in I for I is a graded ideal.

(2) Clearly SI is a left ideal of S. So it is sufficient to show that every homogeneous component of an element of SI is also contained in SI. In fact $\sum x_i a_i = \sum x_i(a^i_{g_1} + \cdots + a^i_{g_j})$ where $a_i \in I$ and $a^i_{g_1} + \cdots + a^i_{g_j}$ is the sum of homogeneous components of a_i. Since each $a^i_{g_s}$ is contained in I (I is a graded ideal) $x_i a^i_{g_s}$ is contained in SI for every i and g_s in G. So every homogeneous component of $\sum x_i a_i$ in S is contained in SI.

PROPOSITION 5. *Let M be a graded left S-module and N be a graded R-submodule of M. Let $N^{x_i} = \{m \in M \mid x_i m \in M\}$. Then N^{x_i} is a graded R-submodule for every $1 \leqq i \leqq n$.*

Proof. At first we know that N^{x_i} is an R-submodule for $x_i(rm) = (x_i r)m = (r' x_i)m = r'(x_i m)N$ for every $r \in R$ and some $r' \in R$. Secondely let $m \in N^{x_i}$ and $m = m_{g_1} + m_{g_2} + \cdots + m_{g_t}$ where $m_{g_i} \in M_{g_i}$. Then $x_i(m_{g_1} + \cdots + m_{g_t}) = x_i m_{g_1} + \cdots + x_i m_{g_t}$ is contained in N. But each $x_i m_{g_j}$ is a homogenious component of $x_i m$ in N. So each $x_i m_{g_j}$ is contained in N for N is graded submodule. Thus m_{g_j} is in N^{x_i}.

2. Essential Extension

In this section, we prove that a result of L.Soueif in a finite normalizing extension of a ring is also true in a graded ring case.

Recall that a graded submodule N is called a graded essential submodule of a graded module M if for every $m \in h(M)$, there exist some $a \in h(R)$ such that $am \in N - \{0\}$ (or equivalently for every graded R-submodule K of M, $K \cap N \neq \{0\}$). We denote N as gr-essential in M. Generally if N is a gr-essential in M, then N is essential in M in usulal meaning. But the converse is not true.

Also recall that a function $f : M \to N$ where both M and N are graded R-modules, say $M = \bigoplus M_g$ and $N = \bigoplus N_g$, is called an R gr-homomorphism if $f = \sum f_{g_i}$ where f_{g_i} is an R- homomorphism and $f_{g_i}(M_g) \subset N_{gg_i}$ (we call f_{g_i} a graded morphism of degree g_i). Immediately we know that the set of all R gr-homomorphisms $Hom_{R_{gr}}(M, N)$ is the direct sum of families of graded morphisms of degree g. In fact

$Hom_{R_{gr}} = \bigoplus_{g\in G} Hom_R(M,N)_g$ where the right hand side is the set of all graded morphisms of degree g. Clearly $Hom_{R_{gr}}(M,N)$ is a graded abelian group of type G. Moreover we get the following lemma.

LEMMA 6. *Let R be graded rings of type G and S be a finite normalizing extension ring of R. If M is a graded right S-module and an $R-S$ bimodule, then $Hom_{R_{gr}}(S,M)$ is a graded left S-module via $(sf)m = f(ms)$*

Proof. Let $f \in Hom_{R_{gr}}(M,N)$ and $s \in S$. At first we show that sf is a sum of some graded morphisms of degree $g_i's$. Let $f = f_{g_1} + f_{g_2} + \cdots + f_{g_n}$ where f_{g_i} is a gradeed morphism of degree g_i and $s = s_{k_1}+s_{k_2}+\cdots+s_{k_t}$ where $s_{k_j} \in S_{k_j}$. Since $f(m) = f_{g_1}(m)+\cdots+f_{g_n}(m)$ for every $m \in M$ and $(s_{k_1}+s_{k_2}+\cdots+s_{k_t})f(m) = f(m(s_{k_1}+\cdots+s_{k_t})) = s_{k_1}f(m)+\cdots+s_{k_t}f(m)$, so $sf = s_{k_1}+s_{k_2}f_{g_1}+\cdots+s_{k_t}f_{g_1}+\cdots+s_{k_j}f_{g_i}+\cdots+s_{k_t}f_{g_n}$. So it is sufficient to show that $s_{k_j}f_{g_i}$ is a graded morphism of degree k_jg_i. In fact $(s_{k_j}f_{g_i})(m_g) = f_{g_i}(m_g s_{k_j}) \subset M_{gk_jg_i}$ for every $g, g_i, k_j \in G$. Also it shows that $S_k Hom_R(M,N)_g$ is contained in $Hom_R(M,N)_{gk}$ for every $g,k \in G$.

Finally we get the following result.

PROPOSITION 7. *If N is a graded R-module and M is a gr- essential submodule of N, then $Hom_{R_{gr}}(S,M)$ is a gr-essential submodule of $Hom_{R_{gr}}(S,N)$ as a graded R-module. Consequently it is gr-essential submodule as a graded S-module.*

Proof. From above lemma we know that $Hom_{R_{gr}}(S,M)$ is a graded left S-module and $Hom_{R_{gr}}(S,N)$ is a graded submodule of $Hom_{R_{gr}}$ (S,M). Let $0 \neq f_g \in Hom_R(S,N)$. We want to show that there exist some $r_k \in R_k$ such that $r_k * f_g \neq 0$ and $(r_k * f_g)(x_i) \in M$ for every x_i where $\{x_i\}$ is a normalizing base for S over R. At first we can show that $f_g(x_1R)$ is a graded R-submodule of N. In fact $Rf_g(x_1R) = f_g(Rx_1RR) \subset f_g(x_1R)$ (If R have an identity 1, then we assume $x_1 = 1$ and clearly $xf_g(x_1R) = f_g(R)$) and $f_g(x_1r) = f_g(x_1(r_{g_1} + \cdots + r_{g_t})) = f_g(x_1r_{g_1} + \cdots + x_1r_{g_2}) = f_g(x_1r_{g_1} + \cdots + x_1r_{g_t})$ implies each homogenious component $f_g(x_1r_{g_i}) = f_g(r'_{g_i}x_1)$ of $f_g(x_1r) \in S_{g_i}$ is contained in $f_g(x_1R)$. Let $L_1 = \{r \in R \mid f_g(x_1r) \in M\} \neq 0$ for M is gr-essential in N. If $f_g(x_1R) = 0$, then clearly L_1 is R itself. Secondly we also know that $f_g(x_2L_1)$ is a graded R-submodule of N

and similarly L_1 is a graded left ideal of R since $f_g(x_1(r_{g_1}+\cdots+r_{g_t}))=$ $f_g(x_1r_{g_1})+\cdots+f_g(x_1r_{g_t})$ implies that $f_g(x_1r_{g_i})$ is contained in M (In fact each $f_g(x_1r_{g_i})$ is homogenious component of an element of M). $L_2=\{r\in L_1 \mid f_g(x_2r)\in M\}$ is a nonzero left graded ideal. By similar method we can get $L_n=\{r\in L_{n-1}\mid f_g(x_nr)\in M\}$ which is a nonzero graded left ideal of R. So for a homogenious component r_k of r in L_n, $(r_k*f_g)(x_i)=f_g(x_ir_k)$ is contained in M and clearly $r_k*f_g\neq 0$. Thus theorem is proved.

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