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ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF NONLINEAR SEMIGROUPS OF LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let T be a mapping of C into itself. T is said to be a Lipschitzian mapping if for each $n \geq 1$ there exists a positive real number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x,\ y \in C$. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \geq 1$ and asymptotically nonexpansive [8] if $\lim_{n\to\infty} k_n = 1$, respectively. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathcal{S}$ is called an asymptotically nonexpansive semigroup on C if $T(0) = I$ (the identity mapping on C), $T(t+s) = T(t)T(s)$ for all $t,\ s \geq 0$, $T(t)x$ is continuous in $t \geq 0$ for each $x \in C$, and there exists a constant $k_t > 0$ with $\lim_{t\to\infty} k_t = 1$ such that

$$\|T(t)x - T(t)y\| \leq k_t \|x - y\|$$

for all $x,\ y \in C$ and $t \geq 0$. If $k_t = 1$ for all $t \geq 0$, then $\mathcal{S}$ is called a nonexpansive semigroup on C.

Baillon [1] (see also Baillon and Brézis [3]) proved the following nonlinear ergodic theorem : If X is Hilbert space and $F(\mathcal{S}) = \{z \in C : T(t)z = z \text{ for all } t \geq 0\} \neq \emptyset$, then for every $x \in C$, $T(t)x$ is weakly almost convergent as $t \to \infty$ to a point y of $F(\mathcal{S})$, i.e.,

$$\text{weak} - \lim_{t\to\infty} \frac{1}{t} \int_0^t T(\tau + h)d\tau = y$$

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uniformly for $h \geq 0$. Baillon [2], Bruck [6] and Reich [16] extended this result to tne case of uniformly convex Banach spaces with Fréchet differentiable norm, and Hirano [11] succeeded in extending it to the case of uniformly convex Banach spaces which satisfying Opial's condition. Recently, Tan and Xu [22] established the nonlinear ergodic theorem for asymptotically nonexpansive semigroups in a uniformly convex Banach space with a Fréchet differentiable norm.

On the other hand, in [6], Bruck introduced the notion of an almost–orbit of a nonexpansive mapping. Miyadera and Kobayashi [14] extended the notion to the case of a nonexpansive semigroup on C and proved the weak almost convergence of such an almost–orbit in a uniformly convex Banach spaces with a Fréchet differentiable norm. Takahashi and Zhang [19, 20] extended the notion to asymptotically nonexpansive semigroups and investigated the weak convergence of such an almost–orbit in a Banach space just above mentioned.

The purpose of this paper is to study the asymptotic behavior of almost–orbits of asymptotically nonexpansive semigroups in a uniformly convex Banach space with a Fréchet differentiable norm. Our results (Theorem 4.1, 4.2 and 4.4) extends some previous results in [3, 6, 10, 12, 14, 16, 21, 22] for nonexpansive mappings or in a Hilbert space setting to asymptotically nonexpansive mappings or in a Banach space setting, because for each $x \in C$, $T(\cdot)x : [0, \infty) \to C$ is an almost–orbit of $\{T(t) : t \geq 0\}$. Section 2 is a preliminary part. In Section 3, we prove several lemmas which are crucial for our study. Main results are given in Section 4. First we provide the weak almost convergence of almost–orbits of asymptotically nonexpansive semigroups. Using the result, we study the weak convergence of an almost–orbit itself. Next we prove the existence of a nonexpansive retraction from the set of almost–orbits of the semigroup S onto $F(S)$.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $(X^*, \|\cdot\|)$ be its dual, that is, tha space of all continuous linear functionals on X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). With each $x \in X$, we associated the set

$$J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$$

Using the Hahn–Banach theorem, it is readily verified that $J(x) \neq \emptyset$ for any $x \in X$. The multi-valued mapping $J : X \to X^*$ is called the duality mapping of X. Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then a Banach space X is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\ y \in U$. In this case, the norm of X is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$.

Let C be a nonempty closed convex subset of X. Then a one-parameter family $\mathcal{S} = \{T(t) : t \geq 0\}$ if mappings from C into itself is said to be a Lipschitzian semigroup on C if $\mathcal{S}$ satisfies the following conditions :

(i) $T(0)x = x$ for $x \in C$;

(ii) $T(t+s)x = T(t)T(s)x$ for $x \in C$ and $t,\ s \geq 0$;

(iii) for each $x \in C$, the mapping $T(t)x$ is continuous for $t \in [0, \infty)$;

(iv) for each $t \geq 0$, there exists a constant $k_t > 0$ such that

$$\|T(t)x - T(t)y\| \leq k_t \|x - y\|$$

for all $x,\ y \in C$. A Lipschitzian semigroup $\mathcal{S}$ is said to be nonexpansive if $k_t = 1$ for all $t \geq 0$ and asymptotically nonexpansive if $\lim_{t \to \infty} k_t = 1$, respectively. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\{T(t) : t \geq 0\}$ in C, i.e.,

$$F(\mathcal{S}) = \bigcap_{t \geq 0} F(T(t))$$

where $F(T(t))$ is the set of fixed points of $T(t)$, i.e., $F(T(t)) = \{x \in C : T(t)x = x\}$. A continuous function $u : [0, \infty) \to C$ is said to be an almost-orbit of $\mathcal{S} = \{T(t) : t \geq 0\}$ of

$$\lim_{t \to \infty} (\sup_{s \geq 0} \|u(s+t) - T(s)u(t)\|) = 0 \tag{2.2}$$

We denote by $\omega_w(u)$ the set of all weak limit points subnets of the net $\{u(t) : t \geq 0\}$. For a subset D of X, $\overline{co}D$ denotes the closure of the convex hull of D.

Finally, we racall the notion of almost convergence due to Lorentz [13].

Definition. Let X be a Banach space. a continuous function $u : [0, \infty) \to X$ is said to be weakly almost convergent to an element y of X if

$$\text{weak} - \lim_{t \to \infty} \frac{1}{t} \int_0^t u(\tau + h) d\tau = y$$

uniformly in $h \geq 0$.

3. Lemmas

In this section, we prepare several lemmas as in [14, 15, 19, 22] which are crucial in studying the asymptotic behavior of almost-orbits.

LEMMA 3.1 [22]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be an asymptotically nonecpansive semigroup on C. Then $F(\mathcal{S})$ is an nonempty closed convex subset of C.*

LEMMA 3.2 [22]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing, convex and continuous function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that for any Lipschitzian mapping $T : C \to X$, any finitely many elements $x_1, \cdots, x_n$ in C and numbers $a_1, \cdots, a_n$ in $[0, \infty)$ with $\sum_{i=1}^n a_i = 1$, there holds the following inequality :*

$$\begin{aligned} &\|T(\sum_{i=1}^n a_i x_i) - \sum_{i=1}^n a_i T x_i\| \\ &\leq L g^{-1}(\max_{1 \leq i,j \leq n} \{\|x_i - x_j\| - \|Tx_i - Tx_j\|\} + (1 - L^{-1})d) \end{aligned} \tag{3.1}$$

where $L \geq 1$ is the Lipschitz constant of T and d is the diameter of C.

LEMMA 3.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be an asymptotically nonecpansive semigroup on C. Then for every net $\{x_t\}_{t\geq 0}$ of elements of C, the conditions $x_t \to x$ weakly as $t \to \infty$ and $\limsup_{s\to\infty}(\limsup_{t\to\infty} \|x_t - T(s)x_t\|) = 0$ imply that $x \in F(\mathcal{S})$, i.e., x is a common fixed point of $\mathcal{S}$.*

Proof. For any arbitrary $\varepsilon > 0$, choose $s_0 \geq 0$ such that

$$\limsup_{t\to\infty} \|x_t - T(s)x_t\| < \varepsilon \tag{3.2}$$

for all $s \geq s_0$. Since $x_t \to x$ weakly, $x \in \overline{co}\{x_h : h \geq t\}$ for each $t \geq 0$ and hence we can choose for each t, a $y_t = \sum_{i=1}^m \lambda_i x_{t_i}$ with $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$, $t_i \geq t$ such that

$$\|x - y_t\| < \frac{1}{t}. \tag{3.3}$$

Now for an arbitrary but fixed $s \geq s_0$, by (3.2) we can find a t_0 such that

$$\|x_t - T(s)x_t\| < \varepsilon \tag{3.4}$$

It then follows from (3.4) and Lemma 3.2 that for $t \geq t_0$

$$\begin{aligned}
&\|T(s)y_t - y_t\| \\
&= \|T(s)(\sum_{i=1}^m \lambda_i x_{t_i}) - (\sum_{i=1}^m \lambda_i x_{t_i})\| \\
&\leq \|T(s)(\sum_{i=1}^m \lambda_i x_{t_i}) - \sum_{i=1}^m \lambda_i T(s)x_{t_i}\| + \sum_{i=1}^m \lambda_i\|T(s)x_{t_i} - x_{t_i}\| \\
&\leq k_s g^{-1}(\max_{1\leq i,j\leq m}\{\|x_{t_i} - x_{t_j}\| - \|T(s)x_{t_i} - T(s)x_{t_j}\|\} \\
&\qquad + (1 - k_s^{-1})d) + \varepsilon.
\end{aligned}$$

where d is the diameter of C. By (3.4), since

$$\begin{aligned}
&\|x_{t_i} - x_{t_j}\| \\
&\leq \|x_{t_i} - T(s)x_{t_i}\| + \|T(s)x_{t_i} - T(s)x_{t_j}\| + \|x_{t_j} - T(s)x_{t_j}\| \\
&\leq 2\varepsilon + \|T(s)x_{t_i} - T(s)x_{t_j}\|,
\end{aligned}$$

we obtain

$$\|T(s)y_t - y_t\| \le k_s g^{-1}((1-k_s^{-1})d + 2\varepsilon) + \varepsilon$$

for all $t \ge t_0$ and hence

$$\limsup_{t\to\infty} \|T(s)y_t - y_t\| \le k_s g^{-1}((1-k_s^{-1})d + 2\varepsilon) + \varepsilon.$$

Therefore

$$\begin{aligned} &\|T(s)x - x\| \\ \le\ &\limsup_{t\to\infty}(\|T(s)x - T(s)y_t\| + \|T(s)y_t - y_t\| + \|y_t - x\|) \\ \le\ &\limsup_{t\to\infty}((1+k_s)\|y_t - x\| + \|T(s)y_t - y_t\|) \\ \le\ &k_s g^{-1}((1-k_s^{-1})d + 2\varepsilon) + \varepsilon \to 0 \quad \text{as } s \to \infty \text{ and } \varepsilon \to 0 \end{aligned}$$

This show that $T(s)x \to x$ strongly as $s \to \infty$ and hence $x \in F(\mathcal{S})$ by continuity of $\mathcal{S}$. The proof is complete.

As a direct consequence, we have a partial extension of a result of Browder [4] to asymptotically nonexpansive mappings.

COROLLARY 3.4. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, i.e., for any net $\{x_t\}$ in C, the condition $x_t \to x$ weakly and $(I - T)x_t \to 0$ strongly imply $(I - T)x = 0$.*

LEMMA 3.5 [15]. *Let C be a nonempty closed convex subset of a Banach space X and let $\mathcal{S} = \{T(t) : t \ge 0\}$ be an asymptotically nonexpansive semigroup on C. If u and v are almost–orbits of $\mathcal{S}$, then $\lim_{t\to\infty} \|u(t) - v(t)\|$ exists. In particular, for every $z \in F(\mathcal{S})$, $\lim_{t\to\infty} \|u(t) - z\|$ exists.*

LEMMA 3.6. *Let C be a nonempty closed convex subset of a Banach space X, $\mathcal{S} = \{T(t) : t \ge 0\}$ an asymptotically nonexpansive semigroup on C, and u an almost–orbit of $\mathcal{S}$. Then for each $h \ge 0$,* *the function $v : [0,\infty) \to C$ defined by $v(t) = T(h)u(t)$ is also an almost-orbit of S.*

Proof. Let $\phi(t) = \sup_{s\geq 0} \|u(s+t) - T(s)u(t)\|$. Since

$$
\begin{aligned}
&\|v(s+t) - T(s)v(t)\| \\
&= \|T(h)u(s+t) - T(s)T(h)u(t)\| \\
&\leq \|T(h)u(s+t) - u(h+s+t)\| + \|u(h+s+t) - T(s+h)u(t)\| \\
&\leq \phi(s+t) + \phi(t),
\end{aligned}
$$

the result follows.

LEMMA 3.7. *Let $S = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup and let u be an almost-orbit of S. Then u is uniformly continuous on $[0,\infty)$.*

Proof. Let $\phi(t) = \sup_{s\geq 0} \|u(s+t) - T(s)u(t)\|$ and let $\varepsilon > 0$ be arbitrary. Then we can choose $t_0 = t_0(\varepsilon) > 0$ such that $\phi(t) < \frac{\varepsilon}{3}$ for $t \geq t_0$. Since u is uniformly continuous on $[0, t_0+1]$, there is a positive $\delta = \delta(\varepsilon) < 1$ such that $\|u(t) - u(t')\| < \frac{\varepsilon}{3M}$ for $t,\ t' \in [0, t_0+1]$ with $|t - t'| < \delta$, where $M = \sup_{t\geq 0} k_t$. Now let $t,\ t' \in [0,\infty)$ be such that $0 < t - t' < \delta$. If $t \leq t_0$, then $t' < t_0 + 1$ and so $\|u(t') - u(t)\| < \frac{\varepsilon}{3M}$¿ If $t > t_0$, then

$$
\begin{aligned}
\|u(t') - u(t)\| &\leq \|u(t') - T(t-t_0)u(t'-t+t_0)\| \\
&\quad + \|T(t-t_0)u(t'-t+t_0) - T(t-t_0)u(t_0)\| \\
&\quad + \|u(t) - T(t-t_0)u(t_0)\| \\
&\leq \phi(t'-t+t_0) + \phi(t_0) + k_{t-t_0}\|u(t'-t+t_0) - u(t_0)\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + M\frac{\varepsilon}{3M} = \varepsilon.
\end{aligned}
$$

Consequently, $\|u(t') - u(t)\| < \varepsilon$ if $|t' - t| < \delta$. This completes the proof.

LEMMA 3.8. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $S = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C, and u an almost-orbit* *of $\mathcal{S}$. Then for each $t > 0$ and $\varepsilon > 0$, there exist $r_t = r_0(t) > 0$ and $h_\varepsilon = h_0(\varepsilon) > 0$ depending only on t and ε respectively such that*

$$\begin{aligned}&\|T(h)(\frac{1}{t}\int_0^t u(r+\tau)d\tau) - \frac{1}{t}\int_0^t u(h+r+\tau)d\tau\| \\ &\leq \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{t}+\varepsilon d) + \frac{1}{t}\end{aligned}$$

for all $r \geq r_t$ and $h \geq h_\varepsilon$, where g is as in Lemma 3.2 and d is the diameter of C.

Proof. By Lemma 3.7, there is a $\delta = \delta(t) > 0$ such that

$$\|u(s) - u(s')\| < \frac{1}{t}$$

whenever $s,\ s' \in [0,\infty)$ such that $|s - s'| < \delta$. We can also choose $r_1(t) > 0$ depending only on t such that

$$\phi(r) = \sup_{h\geq 0}\|u(h+r) - T(h)u(r)\| < \frac{1}{3t} \tag{3.5}$$

for $r \geq r_1(t)$. Choose an integer $N = N(t)$ so large that $N > \frac{t}{\delta(t)}$ and

$$\|\frac{1}{t}\int_0^t u(\tau)d\tau - \frac{1}{t}\sum_{i=1}^{N} u(\tau_i)\Delta\tau\| \leq \frac{1}{t},$$

whenever $\tau_i = (\frac{i}{N})t,\ i = 1,2,\cdots,N$ and $\Delta\tau = \frac{t}{N}$. It follows that

$$\begin{aligned}&\|\frac{1}{t}\int_0^t u(h+r+\tau)d\tau - \frac{1}{t}\sum_{i=1}^{N} u(h+r+\tau_i)\Delta\tau\| \\ &\leq \frac{1}{t}\sum_{i=1}^{N}\int_{\tau_{i-1}}^{\tau_i}\|u(h+r+\tau) - u(h+r+\tau_i)d\tau\| \leq \frac{1}{t}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\|T(h)(\frac{1}{t}\int_0^t u(r+\tau)d\tau) - T(h)(\frac{1}{t}\sum_{i=1}^{N} u(r+\tau_i)\Delta\tau)\| \\ &\leq k_h\|\frac{1}{t}\int_0^t u(r+\tau)d\tau - \frac{1}{t}\sum_{i=1}^{N} u(r+\tau_i)\Delta\tau\| \leq \frac{k_h}{t}.\end{aligned} \tag{3.7}$$

From (3.5), (3.6), (3.7) and Lemma 3.2, it follows that
(3.8)

$$\begin{aligned}
&\|T(h)(\frac{1}{t}\int_0^t u(r+\tau)d\tau) - \frac{1}{t}\int_0^t u(h+r+\tau)d\tau\| \\
&\le \frac{1+k_h}{t} + \|T(h)(\frac{1}{t}\sum_{i=1}^{N} u(r+\tau_i)\Delta\tau - \frac{1}{t}\sum_{i=1}^{N} T(h)u(r+\tau_i)\Delta\tau\| \\
&\quad + \frac{1}{t}\sum_{i=1}^{N}\|T(h)u(r+\tau_i) - u(h+r+\tau_i)\|\Delta\tau \\
&\le \frac{1+k_h}{t} + k_h g^{-1}(\max_{1\le i,j\le N}\{\|u(r+\tau_i) - u(r+\tau_j)\| \\
&\quad - \|T(h)u(r+\tau_i) - T(h)u(r+\tau_j)\|\} + (1-k_h^{-1})d) + \frac{1}{3t} \\
&\le \frac{1+k_h}{t} + k_h g^{-1}(\max_{1\le i,j\le N}\{\|u(r+\tau_i) - u(r+\tau_j)\| \\
&\quad - \|u(h+r+\tau_i)\| - u(h+r+\tau_j)\| + \frac{2}{3t}\} + (1-k_h^{-1})d) + \frac{1}{t}
\end{aligned}$$

for $r \ge r_1(t)$. Since $u(t+h)$ is also an almost-orbit of $\mathcal{S}$ for every $h \ge 0$, by Lemma 3.5, $\lim_{r\to\infty}\|u(r+\tau_i) - u(r+\tau_j)\|$ exists for $i,\ j = 1, 2, \cdots, N$. Then we choose an $r_2(t) > 0$ depending only on t such that

$$\text{(3.9)}\quad \|u(r+\tau_i) - u(r+\tau_j)\| - \|u(h+r+\tau_i) - u(h+r+\tau_j)\| < \frac{1}{3t}$$

for $r > r_2(t)$ and $i,\ j = 1, 2, \cdots, N$. Let $r_t = \max\{r_1(t), r_2(t)\}$. Finally, we choose an $h_\varepsilon > 0$ depending only on ε such that

$$\text{(3.10)}\qquad k_h < 1+\varepsilon$$

for all $h \ge h_\varepsilon$. From (3.8), (3.9) and (3.10), it follows that

$$\begin{aligned}
&\|T(h)(\frac{1}{t}\int_0^t u(r+\tau)d\tau) - \frac{1}{t}\int_0^t u(h+r+\tau)d\tau\| \\
&\le \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{3t} + \frac{2}{3t} + \frac{\varepsilon}{1+\varepsilon}d) + \frac{1}{t} \\
&\le \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{t} + \varepsilon d) + \frac{1}{t}
\end{aligned}$$

for all $r \ge r_t$ and $h \ge h_\varepsilon$. The proof is complete.

LEMMA 3.9 [15]. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C a nonempty bounded closed convex subset of X, $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C, and u an almost–orbit of $\mathcal{S}$. Then*

$$F(\mathcal{S}) \cap \bigcap_{s \geq 0} \overline{co}\{u(t) : t \geq s\}$$

is at most a singleton.

By Lemma 3.1, the nearest point projection P of X onto $F(\mathcal{S})$ is well–defined. Finally, we prepare the following lemma whose result (i) is given in [19]. For the study and completeness, we give the detail proof.

LEMMA 3.10. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C, u an almost–orbit of $\mathcal{S}$, and P the nearest pont projection of X onto $F(\mathcal{S})$. Then we have the follwing :*

(i) *$\{Pu(t)\}$ converges strongly to some z_0, where z_0 is a unique element in $F(\mathcal{S})$ such that*

$$\lim_{t\to\infty} \|u(t) - z_0\| = \min\{\lim_{t\to\infty} \|u(t) - z\| : z \in F(\mathcal{S})\}.$$

(ii) *Moreover, if X is smooth and $\{u(t)\}$ converges weakly to a point $y \in F(\mathcal{S})$, then $y = z_0$.*

Proof. (i) By Lemma 3.5, we know that $\lim_{t\to\infty} \|u(t) - z\| = f(z)$ exists for each $z \in F(\mathcal{S})$. Let $R = \inf\{f(z) : z \in F(\mathcal{S})\}$ and $M = \{w \in F(\mathcal{S}) : f(w) = R\}$. Then, since $f(z)$ is convex and continuous on $F(\mathcal{S})$ and $f(z) \to \infty$ as $\|z\| \to \infty$, M is a nonempty closed convex bounded subset of $F(\mathcal{S})$. Fix $z_0 \in M$ with $f(z_0) = R$. Since P is the nearest point projection of X onto $F(\mathcal{S})$, we have $\|u(t) - Pu(t)\| \leq \|u(t) - y\|$ for $t \geq 0$ and $y \in F(\mathcal{S})$, and hence

$$\lim_{t\to\infty} \|u(t) - Pu(t)\| \leq R.$$

Suppose that $\lim_{t\to\infty}\|u(t)-Pu(t)\|<R$. Then we may choose $\varepsilon>0$ and $t_0\geq 0$ so that $\|u(t)-Pu(t)\|\leq R-\varepsilon$ for all $t\geq t_0$. Since

$$\begin{aligned}\|u(t+s)-Pu(s)\| &\leq \|u(t+s)-T(t)u(s)\|+\|T(t)u(s)-Pu(s)\|\\ &\leq \phi(s)+k_t\|u(s)-Pu(s)\|\end{aligned}$$

for all $t,\ s\geq 0$ and $\lim_{s\to\infty}\phi(s)=0$, where $\phi(s)=\sup_{t\geq 0}\|u(t+s)-T(t)u(s)\|$, we can choose $s\geq t_0$ such that

$$\|u(t+s)-Pu(s)\|\leq k_t\|u(s)-Pu(s)\|+\frac{\varepsilon}{2}\leq k_t(R-\varepsilon)+\frac{\varepsilon}{2}$$

for all $t\geq 0$. Therefore we can obtain that

$$\begin{aligned}\lim_{t\to\infty}\|u(t)-Pu(t)\| &\leq \lim_{t\to\infty}k_t(R-\varepsilon)+\frac{\varepsilon}{2}\\ &=R-\varepsilon+\frac{\varepsilon}{2}=R-\frac{\varepsilon}{2}<R.\end{aligned}$$

This is a contradiction. So, we conclude that

$$\lim_{t\to\infty}\|u(t)-Pu(t)\|=R.$$

Now we claim that $\lim_{t\to\infty}Pu(t)=z_0$. If not, then there exists $\varepsilon>0$ such that for any $t\geq 0$ $\|Pu(t')-z_0\|\geq\varepsilon$ for some $t\geq t'$. Choose $a>0$ so smallthat

$$(R+a)(1-\delta(\frac{\varepsilon}{R+a}))=R_1<R,$$

where δ is the modulus of convexity of X (see [9]). We have $\|u(t')-Pu(t')\|\leq R+a$ and $\|u(t')-z_0\|\leq R+a$ for large enough t'. Therefore we have

$$\|u(t')-\frac{Pu(t')+z_0}{2}\|\leq(R+a)(1-\delta(\frac{\varepsilon}{R+a}))=R_1.$$

Since $w_{t'}=\frac{Pu(t')+z_0}{2}\in F(\mathcal{S})$, as in the above

$$\|u(t+t')-w_{t'}\|\leq k_t\|u(t')-w_{t'}\|+\phi(t')$$

for all $t \geq 0$. Since $\lim_{s\to\infty}\phi(s) = 0$, there is a $t' \geq 0$ such that

$$\|u(t+t') - w_{t'}\| \leq k_t\|u(t') - w_{t'}\| + \frac{R - R_1}{2} \leq k_t R_1 + \frac{R - R_1}{2},$$

and hence

$$\begin{aligned}\lim_{t\to\infty}\|u(t) - w_{t'}\| &\leq (\lim_{t\to\infty} k_t)R_1 + \frac{R - R_1}{2} \\ &= R_1 + \frac{R - R_1}{2} = \frac{R + R_1}{2} < R.\end{aligned}$$

This contradicts the fact $R = \inf\{f(z) : z \in F(\mathcal{S})\}$. Therefore we have $\lim_{t\to\infty} Pu(t) = z_0$. Consequently, it follows that the element $z_0 \in F(\mathcal{S})$ with $f(z_0) = \min\{g(z) : z \in F(\mathcal{S})\}$ is unique.

(ii) Now suppose that $\{u(t)\}$ converges weakly to a $y \in F(\mathcal{S})$. Then, since $(u(t) - Pu(t), J(f - Pu(t)) \leq 0$ for all $f \in F(\mathcal{S})$ (see [9]), by taking the limit as $t \to \infty$, we get $(y - z_0, J(f - z_0) \leq 0$ for all $f \in F(\mathcal{S})$. Because $y \in F(\mathcal{S})$, we in particular obtain $\|y - z_0\|^2 = (y - z_0, J(y - z_0)) \leq 0$ and hence $y = z_0$. This completes the proof.

4. Asymptotic behavior

In this section, we prove the nonlinear ergodic theorem for almost–orbits of asymptotically nonexpansive semigroups in Banach spaces.

THEOREM 4.1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C a nonempty bounded closed convex subset of X, and $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. Then every almost–orbit u of $\mathcal{S}$ is weakly almost convergent to the unique point of the set $F(\mathcal{S}) \cap \bigcap_{s\geq 0} \overline{co}\{u(t) : t \geq 0\}$.*

Proof. By Lemma 3.8, we have

$$\begin{aligned}&\|T(h)(\frac{1}{t}\int_0^t u(r_t + \tau)d\tau) - \frac{1}{t}\int_0^t u(h + r_t + \tau)d\tau\| \\ &\leq \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{t} + \varepsilon d) + \frac{1}{t}\end{aligned}$$

for all $t > 0$ and $h \geq h_\varepsilon$, where r_t and h_ε are as in Lemma 3.8, and d is the diameter of C. Noting that

$$\begin{aligned}
&\|\frac{1}{t}\int_0^t u(h+r_t+\tau)d\tau - \frac{1}{t}\int_0^t u(r_t+\tau)d\tau\| \\
&= \|\frac{1}{t}\int_0^h (u(t+r_t+\tau) - u(r_t+\tau))d\tau\| \\
&\leq \frac{h}{t}d,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|T(h)(\frac{1}{t}\int_0^t u(r_t+\tau)d\tau) - \frac{1}{t}\int_0^t u(r_t+\tau)d\tau\| \\
&\leq \|T(h)(\frac{1}{t}\int_0^t u(r_t+\tau)d\tau) - \frac{1}{t}\int_0^t u(h+r_t+\tau)d\tau\| \\
&\quad + \|\frac{1}{t}\int_0^t u(h+r_t+\tau)d\tau - \frac{1}{t}\int_0^t u(r_t+\tau)d\tau\| \\
&\leq \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{t}+\varepsilon d) + \frac{1+hd}{t}
\end{aligned}$$

for all $h \geq h_\varepsilon$ and $t > 0$. Therefore we have

$$\limsup_{h\to\infty}(\limsup_{t\to\infty}\|T(h)(\frac{1}{t}\int_0^t u(r_t+\tau)d\tau) - \frac{1}{t}\int_0^t u(r_t+\tau)d\tau\|) = 0.$$

It follows from Lemma 3.3 that $\{\frac{1}{t}\int_0^t u(r_t+\tau)d\tau\}$ converges weakly to y, the unique point of the set $F(\mathcal{S}) \cap \bigcap_{s\geq 0}\overline{co}\{u(t) : t \geq s\}$. By the same method, we also obtain

$$\begin{aligned}
&\|T(h)(\int_0^t u(r_t+p_t+\tau)d\tau) - \frac{1}{t}\int_0^t u(r_t+p_t+\tau)d\tau\| \\
&\leq \frac{2+\varepsilon}{t} + (1+\varepsilon)g^{-1}(\frac{1}{t}+\varepsilon d) + \frac{1+hd}{t}
\end{aligned}$$

for all $t > 0$ and $h \geq h_\varepsilon$, and thus

$$\begin{aligned}
\limsup_{h\to\infty}(\limsup_{t\to\infty}&\|T(h)(\frac{1}{t}\int_0^t u(r_t+p_t+\tau)d\tau) \\
&- \frac{1}{t}\int_0^t u(r_t+p_t+\tau)d\tau\|) = 0.
\end{aligned}$$

It follows again from Lemma 3.3 that $\{\frac{1}{t}\int_0^t u(r_t+p_t+\tau)d\tau\}$ converges weakly to y. This implies that

$$\frac{1}{t}\int_0^t u(r_t+p+\tau)d\tau \longrightarrow y \quad \text{weakly as} \quad t\to\infty$$

uniformly in $p\geq 0$. Now for t and $s\geq r_t$, we have

$$\begin{aligned}
&\frac{1}{s}\int_0^s u(p+\tau)d\tau \\
&=\frac{1}{s}\{\int_0^{r_t} u(p+\tau)d\tau+\int_{r_t}^{qt+r_t} u(p+\tau)d\tau+\int_{qt+r_t}^{s} u(p+\tau)d\tau\} \\
&=\frac{1}{s}\{\int_0^{r_t} u(p+\tau)d\tau+\sum_{j=0}^{q-1}\int_0^{t} u(r_t+jt+p+\tau)d\tau \\
&\qquad\qquad +\int_{qt+r_t}^{s} u(p+\tau)d\tau\},
\end{aligned}$$

where $s=qt+r_t+r,\ r<t$. Since

$$\frac{1}{t}\int_0^t u(r_t+jt+p+\tau)d\tau \longrightarrow y \quad \text{weakly as} \quad t\to\infty$$

uniformly in $p,\ j\geq 0$, we conclude from above that

$$\frac{1}{s}\int_0^s u(p+\tau)d\tau \longrightarrow y \quad \text{weakly as} \quad s\to\infty$$

uniformly in $p\geq 0$. This completes the proof.

Recall that $\{u(t)\}$ is said to be weakly asymptotically regular if

$$\lim_{t\to\infty}(u(t+h)-u(t))=0$$

weakly for all $h\geq 0$. Now we discuss the weak convergence of an almost-orbit $\{u(t)\}$ itself.

THEOREM 4.2. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C a nonemprty bounded closed convex subset of X, and $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. Then for each almost-orbit u of $\mathcal{S}$, the following statements are equivalent :*

(i) *$\{u(t)\}$ converges weakly to a common fixed point of $\mathcal{S}$;*

(ii) *$\{u(t)\}$ is weakly asymptotically regular.*

Proof. The fact that (i) $\Longrightarrow$ (ii) is obvious.

(i) $\Longleftarrow$ (ii). The result follows from Theorem 4.1 and the fact that weak $\lim_{t\to\infty}(u(t+h)-u(t)) = 0$ for all $h \geq 0$ is a Tauberian condition for weak almost convergence of $\{u(t)\}$ (cf. [13, 18]).

COROLLARY 4.3. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C a nonempty bounded closed convex subset of X and $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. If for each almost-orbit u of $\mathcal{S}$, $\{u(t)\}$ is weakly asymptotically regular, then $\{u(t)\}$ converges weakly to $y \in F(\mathcal{S})$ and $y = \lim_{t\to\infty} Pu(t)$, where P is the nearest point projection of X onto $F(\mathcal{S})$.*

Proof. The result follows from Lemma 3.10 and Theorem 4.2.

Finally, we prove the existence of a nonexpansive retraction for almost-orbits of an asymptotically nonexpansive semigroup.

THEOREM 4.4. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C an nonempty bounded closed convex subset of X, and $\mathcal{S} = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. Then the limit*

$$Qu = \text{weak} - \lim_{t\to\infty} \frac{1}{t}\int_0^t u(\tau)d\tau$$

is the unique retraction from the set $AO(\mathcal{S})$ of all almost-orbits of $\mathcal{S}$ onto $F(\mathcal{S})$ such that

(i) *Q is nonexpansive in the sense that*

$$\|Q(u) - Q(v)\| \leq \|u - v\|_\infty = \sup_{t\geq 0}\|u(t) - v(t)\|$$

for $u,\ v \in AO(\mathcal{S})$;

(ii) $QT(h)u = T(h)Qu = Qu$ *for* $u \in AO(\mathcal{S})$ and $h \geq 0$;

(iii) $Qu \in \bigcap_{s\geq 0} \overline{co}\{u(t) : t \geq s\}$ *for* $u \in AO(\mathcal{S})$.

Proof. (i) By Lemma 3.1, the fact $F(\mathcal{S}) \subset AO(\mathcal{S})$, and Theorem 4.1, the retraction Q from $AO(\mathcal{S})$ onto $F(\mathcal{S})$ is well defined. Also, by the weak lower semicontinuity of the norm of X, it follows from Theorem 4.1 that for each $h \geq 0$

$$\begin{aligned}\|Qu - Qv\| &\leq \liminf_{t\to\infty} \Big\|\frac{1}{t}\int_0^t u(h+\tau)d\tau - \frac{1}{t}\int_0^t v(h+\tau)d\tau\Big\| \\ &\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \|u(h+\tau) - v(h+\tau)\|d\tau \\ &\leq \|u - v\|_\infty,\end{aligned}$$

that is, Q is nonexpansive.

(ii) Let $u \in AO(\mathcal{S})$ and $h \geq 0$. First we observe that $QT(h)u$ is well defined by Lemma 3.6. Since $Qu \in F(\mathcal{S})$, we have $T(h)Qu = Qu$. Thus it remains to prove $QT(h)u = Qu$. In fact, we have

$$\begin{aligned}QT(h)u &= \text{weak} - \lim_{t\to\infty} \frac{1}{t}\int_0^t T(h)u(\tau)d\tau \\ &= \text{weak} - \lim_{t\to\infty} \frac{1}{t}\int_0^t u(h+\tau)d\tau \\ &\qquad + \text{weak} - \lim_{t\to\infty} \frac{1}{t}\int_0^t (T(h)u(\tau) - u(h+\tau))d\tau \\ &= \text{weak} - \lim_{t\to\infty} \frac{1}{t}\int_0^t u(h+\tau)d\tau = Qu,\end{aligned}$$

since

$$\frac{1}{t}\int_0^t \|T(h)u(\tau) - u(h+\tau)\| \leq \frac{1}{t}\int_0^t \phi(\tau)d\tau \to 0 \ \text{ as } \ t \to \infty,$$

where $\phi(t) = \sup_{s\geq 0} \|u(s+t) - T(s)u(t)\| \to 0$ as $t \to \infty$.

Finally, (iii) and uniqueness of such a retraction Q follows from Lemma 3.19 and Theorem 4.1. The proof is complete.

REMARK. (1) Since for each $x \in C$, $T(\cdot)x : [0, \infty) \to C$ is an almost-orbit of $\{T(t) : t \geq 0\}$, Theorem 4.1 and 4.2 are improvements of Theorem 3.1 and 3.2 in [22].

(2) Our results (Theorem 4.1 and 4.2, and Corollary 4.3) are also generalizations of the corresponding results in [3, 6, 10, 12, 14, 16, 21] for nonexpansive mappings or in a Hilbert space setting to asymptotically nonexpansive mappings or in a Banach space setting. In particular, the result $y = \lim_{t \to \infty} Pu(t)$ in Corollary 4.3 is of interest.

(3) Theorem 4.4 seems to be new even for nonexpansive semigroups. If $u(t) = T(t)x$ and we define $Qx = \text{weak} - \lim_{t \to \infty} \frac{1}{t} \int_0^t T(\tau)x d\tau$ for each $x \in C$ as in [15, 22], we also obtain the unique nonexpansive retraction Q from C onto $F(S)$ such that

(i) $QT(t)x = T(t)Qx = Qx$ for $x \in C$ and $t \geq 0$,

(ii) $Qx \in \bigcap_{s \geq 0} \overline{co}\{T(t)x : t \geq s\}$ for $x \in C$.

(4) We do not know whether our results are valid for the orbit $\{T(t)x : t \geq 0\}$ at x of a strongly measurable (not necessarily continous) asymptotically nonexpansive semigroup $\mathcal{S} = \{T(t) : t \geq 0\}$ on C (see [18, p. 550]).

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