FLOER HOMOLOGY AND COHOMOLOGY GROUP OF THE SPHERE BUNDLES

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1.Introduction

On a finite dimensional manifold a Morse-Smale function induces Morse-Smale gradient flows between the critical points and connecting orbits as boundary maps on the chain complex. The complex gives rise to the usual singluar cohomology(homology) on the manifold. A Morse-Smale gradient flow plays an essential role in Floer's work. We will use this technique to study the topology on the sphere bundle using the Morse theory on the basemanifold and Euler class of the sphere bundle.

We introduce the Floer homology group and the Thereom of Floer for the Arnold conjecture. Floer used the infinite dimensional version of Morse theory to prove Arnold conjecture that the number of fixed points of an exact symplectic diffeomorphism on a symplectic manifold can be estimated below by the sum of the Betti numbers if the fixed points are nondegenerate. He defined a relative index for a pair of critical points and generalized the Morse complex of critical points and connecting orbits to the infinite dimensional situation of the loop space which led to the concept of Floer homology.

In this paper we will outline the main ideas of Floer's proof of the Arnold conjecture and at some places suggest slight modifications. We construct a chain complex whose cohomolgy is isomorphic to the cohomology of the total space of the sphere bundle. Using a Morse-Smale function on the base manifold and a generic section on the vector bundle we define a Morse-Smale fuction on the total space of the sphere bundle.

2. Preliminaries and Basic Theorems

Let M be a smooth compact Remannian manifold of dimension n. We assume that $f : M \to M$ is a Morse fuction on M. For a point $x \in M$ let α_x be the flow line through x. Then $\frac{d\alpha_x(t)}{dt} + \nabla_{\alpha_x(t)}(f) = 0$ and the initial condition $\alpha_x(0) = x$. Let a be a critical point of f and index of f at a is k. We define the manifold $W^s(a)$ and the unstable manifold $W^u(a)$ as follows ;

$$W^s(a) = \{x \in M \mid \lim_{t \to +\infty} \alpha_x(t) = a\} \cong D^{n-k}$$

$$W^u(a) = \{x \in M \mid \lim_{t \to -\infty} \alpha_x(t) = a\} \cong D^k$$

A Morse fuction $f : M \to R$ is to satisfy the Morse-Smale condition if for any two critical points x and y the unstable and stable manifold $W^u(x)$ and $W^s(y)$ intersect trnsversally.

If $f : M \to R$ is a Morse-Smale fuction, then there is a CW-complex $C(f)$ whose cells correspond to the unstable manifolds of the critical points of f such that $C(f)$ is homotopy equivalent to the manifold M. The CW-complex $C(f)$ is constructed by the Thom-Pontryagin framed submanifolds as follows; Let $M(x, y) = W^u(x) \cap W^s(y)$, where x and y are critical points of f. If $p = index(x)$ and $q = index(y)$, then $M(x, y)$ is a $(p - q)$-dimensional submanifold of M. In this case the number $p - q$ is called the relative index of x and y. In fact $M(x, y)$ is the space of all points that lie on flow lines starting from x and ending at y. There is a natural free action of the real line R on the space $M(x, y)$ given by the flow of $\nabla(f)$. That is, $M(x, y) \times R \to M(x, y)$ is given by $(z, t) \mapsto \alpha_z(t)$, where α_z is the unique flow through x satisfying $\alpha_z(0) = 0$ and $\alpha(t) = -\nabla_{\alpha(t)}(f)$.

If we choose any point c between $f(x)$ and $f(y)$ and set $M(x, y)^c = M(x, y) \times f^-(c)$, then this action restricts to give a diffemorphism $M(x, y)^c \times R \to M(x, y)$. Therefore we have the quotient orbit space $\mathcal{M}(x, y) = M(x, y)/R$ which is called the module space of flow lines from x to y. Furthermore the composition

$$M(x, y)^c \to M(x, y) \to M(x, y)/R \cong \mathcal{M}(x, y)$$

is a diffeomorphism for any value c between $f(x)$ and $f(y)$, and hence $\mathcal{M}(x,y)$ is a manifold of dimensional $p - q - 1$.

The CW-complex $C(f)$ yields the associated cellular-chain complex, which is called the Morse-Smale chain complex

$$\cdots \rightarrow C_k \xrightarrow{\partial_k} C_{k-1} \rightarrow \cdots \xrightarrow{\partial_1} C_0 \rightarrow 0$$

where C_k is the free abelian group generated by the cells of $C(f)$ of dimensional k, the critical points of f of index k. The boundary homomorphisms are determine by the relative attaching maps of k-dimensional skeleton of $C(f)$. We can compute the boundary homomorphisms as follows. Suppose x and y are critical points of relatives index one. Let $ind(x) = p$ and $ind(y) = p - 1$. The space of flows $(\mathcal{M}(x,y), \alpha)$ is a zero dimensional, framed, compact manifold. $\mathcal{M}(x,y)$ is a finite set of points(flow lines) with signs attached to them induced by the framing. Let $n(x,y) \in \mathbf{Z}$ denote the signed number of flow lines, $n(x,y) = \sum \alpha(\gamma)$, where $\gamma \in \mathcal{M}(x,y)$ and $\alpha(\gamma) = \pm 1$ is the sign associated to the flow line γ by the framing α. $n(x,y) \in \mathbf{Z} = \Pi_{p-1}(S^{p-1})$ is the integer given by the degree of the relative attaching map $\phi_{x,y} : S^{p-1} \rightarrow S^{p-1}$.

PROPOSITION 2.1. *The coefficient of* $[y] \in C_{p-1}$ *of the boundary* $\partial_p(x)$ *is given by the formula,* $< \partial_p(x), y > = n(x,y) \in \mathbf{Z}, \partial_p(x) = \sum_y n(x,y)y$ *where the sum runs over all critical points of index* $p - 1$.

We can extend the Morse-Smale chain complex to coefficients in any abelian group G by defining $C_k(G) = C_k \otimes_{\mathbf{Z}} G$ and $\partial_p(G) = \partial_p \otimes I_G : C_p(G) \rightarrow C_{p-1}(G)$.

From above construction we have the following significant result.

(1) $\partial^2(G) = 0$
(2) $H_p(M; G) = \frac{\ker \partial_{p-1}(G)}{\operatorname{Im} \partial_p(G)}$

3. Floer Homology and Arnold Conjecture

The Arnold conjecture states that minimal number of fixed points of an exact simplectomorphism on a symplectic manifold is the sum

of Betti numbers provided that the fixed points are nondegenerate. This was proved by Floer under the assumption that over $\pi_2(M)$ the cohomology class of ω agrees up to a constant with the first Chern class $c_1 \in H^2(M)$ of TM which is regarded as a complex vector bundle via an almost complex structure. He used a Morse type index theory for an indefinite function on the loop space and a relative index for two critical points with infinite Morse index. We will outline the main ideas of Floer's proof. In order to avoid additional difficulities we assume that the integral of ω vanishes over every sphere, i.e.

$$\int_{S^2} u^*\omega = 0, \quad u : S^2 \to M$$

Let (M, ω) be a compact $2n$-dimensional symplectic manifold meaning that $\omega \in \Omega^2(M)$ is a nondegenerate closed 2-form. A symplectomorphism of M is a diffeomorphism $\psi \in Diff(M)$ satisfying $\psi^*\omega = \omega$. It is called exact (or homologous to the indentify) if it can be interpolated by a time dependent Hamiltonian differential equation.

$$(3.1) \qquad\qquad \dot{x}(t) = X_H(x(t), t)$$

Here $H : M \times R \to R$ is a smooth function satisfying $H(x, t+1) = H(x, t)$ and the associated Hamiltonian vector field $X_H : M \times R \to TM$ is defined by

$$\omega(X_H(x, t), \zeta) = -d_x H(x, t)\zeta, \quad \zeta \in T_x M$$

The solutions $x(t)$ of (3.1) determine a 1-parameter family of symplectomorphisms $\psi_t \in Diff(M)$ satisfying $\psi_t(x(0)) = x(t)$ and any symplectomorphism $\psi = \psi_1$ which can be generated this way is called exact. We denote by

$$\mathcal{P}_0 = \{x : R \to M | x\,\text{satisfies}\,(3.1), x(t+1) = x(t), x\,\text{is null-homotopic}\}.$$

the space of contractible 1-periodic solutions of (3.1). A periodic solution $x \in \mathcal{P}_0$ is called nondegenerate if $\det(I - d\psi_1(x(0))) \neq 0$.

THEOREM 3.2. *(Floer) Suppose that $\int_{S^2} u^*\omega = 0$ for any $u : S^2 \to M$ and the contractible 1-periodic solutions of (3.1) are nondegenerate. Then the minimal number of fixed points of the exact symplectomorphism on M is the sum of the Betti numbers of M.*

Let $L(M)$ be the loop space of M and $L_0(M) \subset L(M)$ the subspace of contractible loops. We represent a loop in M by a perodic map $\gamma : R \to M$ satisfying $\gamma(t + 1) = \gamma(t)$.

We define a function on $L_0(M)$ by $f_H : L_0(M) \to R$,

$$f_H(\gamma) = -\int_{D^2} u^*\omega + \int_0^1 H(\gamma(t), t)dt$$

where D^2 is the unit disk and $u : D^2 \to M$ is a smooth function satisfying $u(e^{2\pi i t}) = \gamma(t)$.

PROPOSITION 3.3.

(1) *The fuction f_H is well-defined.*
(2) *If $\gamma \in L_0(M)$ is a critical point, then γ is a solution of $\omega(\gamma, \zeta) = -dH(\gamma, t)\zeta$ for $\zeta \in \Gamma(TM)$.*

Proof.

(1) Since $\gamma \in L_0(M)$ the loop γ is contractible, and hence there is a map $u : D^2 \to M$. Since for any map $v : S^2 \to M$, $\int_{S^2} v^*u = 0$, the integral $\int_{D^2} u^*\omega$ is independent of the choice of u.

(2) The tangent space $T_\gamma L_0(M)$ is represented by the space of vector field $\zeta \in \Gamma(\gamma^*TM)$ along γ satisfying $\zeta(t + 1) = \zeta(t)$ since $\gamma(t + 1) = \gamma(t)$. More precisely let $\Phi : R \times (-\epsilon, \epsilon) \to M$ be given by $\Phi(t, s) = \gamma_s(t)$, $\gamma_0(t) = \gamma(t)$ and

$$\frac{d}{ds}\big|_{s=0}\Phi(t, s) = \zeta(\gamma(t)) \in T_{\gamma(t)}M.$$

So $(\gamma^*\zeta)(t) \in (\gamma^*TM)_t$ and $\gamma^*\zeta \in \Gamma(\gamma^*TM)$, $df_H(\gamma) =$

$T_{\gamma(t)}L_0(M) \to R$, for any $\zeta \in TL_0(M)$.

$$
\begin{aligned}
df_H(\gamma)(\zeta) &= \frac{d}{ds}|_{s=0} f_H(\gamma_s) \\
&= \frac{d}{ds}|_{s=0}[-\int_{D^2} u^*\omega + \int_0^1 H(\gamma_s(t),t)dt] \\
&= -\int_{D^2} \frac{d}{ds}|_{s=0}(u^*\omega) + \int_0^1 (\frac{d}{ds}|_{s=0}H(\gamma_s(t),t))dt \\
&= -\int_{D^2} d[(\frac{d}{ds}|_{s=0}u^*)^*\omega] + \int_0^1 dH(\gamma,t)\zeta dt \\
&= \int_0^1 \omega(\dot{\gamma},\zeta)dt + \int_0^1 dH(\gamma,t)\zeta dt \\
&= \int_0^1 [\omega(\dot{\gamma},\zeta) + dH(\gamma,t)\zeta]dt
\end{aligned}
$$

In order to determine the gradient of f_H we choose an almost complex structure on M meaning an endomorphism $J \in C^\infty(End(TM))$ such that
$J^2 = -1$ and $< \zeta,\eta >= \omega(\zeta, J(x)\eta)$, $\zeta,\eta \in T_xM$ defines a Remannian metric on M. In fact J is an isometry and T_xM is a complex vector space by $z\zeta = s\zeta + tJ(x)$ for $z = s + it \in \mathbf{C}$. A holomorphic curve is a solution $u : S \to M$ of the nonlinear Cauchy-Riemann equations

$$
\bar{\partial}u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0
$$

defined on a Riemann surface S. Holomorphic curves plays an essential role in Floer's work.

PROPOSITION 3.4. *If u is a holomorphic curve, then*

$$
\int_S u^*\omega = \frac{1}{2}\int_S |\nabla u|^2
$$

Let $\nabla H : M \times R \to TM$ be the gradient of H with respect to the x-variable then the associated Hamiltonian vectorfield can be written as $X_H(x,t) = J(x)\nabla H(x,t)$.

Now the gradient of f_H with respect to the induced metric on $L_0(M)$ is given by

$$\nabla f_H(\gamma) = J(\gamma)\dot{\gamma} + \nabla H(\gamma, t) \in T_\gamma L_0(M).$$

A gradient flow line of f_H is a smooth map $u : R \times S^1 \to M$ satisfying

$$(3.5) \qquad \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u, t) = 0$$

At any critical point the Morse index for both f_H and $-f_H$ may be infinite. Nevertheless we can do Morse theory for f_H by studying only the space of bounded solutions, an idea which goes back to C.Conley.

In order to describe the space of bounded solutions of (3.5) we choose any two periodic solutions $x \in \mathcal{P}_0$ and $y \in \mathcal{P}_0$ and denote by $\mathcal{M}(y, x)$ the space of connecting orbits with respect to the gradient flow of f_H. If $u \in \mathcal{M}(y, x)$, then u satisfies (3.5) and $\lim_{s \to -\infty} u(s, t) = y(t)$, $\lim_{s \to \infty} u(s, t) = x(t)$ and the minimize the energy functional

$$\Phi_H(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 (|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_H(u, t)|^2)dtds$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 (|\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u, t)|^2 dtds$$

$$+ f_H(y) - f_H(x).$$

Let $\mathcal{M} = \{u : R \times S^1 \to M | u \text{ satisfies}(3.5), \Phi_H(u) < \infty, u \text{ is null-homotopy}\}$ be the space of solution u of (3.5) along which the decreasing function $f_H(u_s)$ with $u_s(t) = u(s, t)$ is bounded.

THEOREM 3.5. *(Floer)*

(1) $\mathcal{M} = \cup_{x, y \in \mathcal{P}_0} \mathcal{M}(y, x)$.

(2) $\int_S u^*\omega = 0$ for $u : S^2 \to M$ then $\mathcal{M}$ is compact.

Floer used the spaces $\mathcal{M}(y, x)$ of connecting orbits in order to define a relative Morse index.

THEOREM 3.6. *(Floer) The boundary operator satisfies $\partial^F \cdot \partial^F = 0$. Moreover, the boundary groups*

$$H_k^F(M; G) = \frac{\ker \partial_{k-1}^F(G)}{\operatorname{Im}\partial_k^F(G)}$$

are independent of Hamiltonian H and the almost complex structure J used to define them. They agree with the singular homology groups of M.

THEOREM 3.7. *Suppose that $\int_{S^2} u^*\omega = 0$, $u : S^2 \to M$ holds and the contractible periodic solutions of $\dot{x}(t) = X_H(x(t), t)$ with integer period nondegenerate. Then there are infinitely many of them.*

4. Morse Theory on Sphere Bundles

Let E be a vector bundle of rank n with projective map $\pi : E \to M$. Suppose that E is oriented and has a Riemannian structure. We restrict π to the $(n-1)$-sphere bundle $\pi : S(E) \to M$. There is a long exact sequence.

THEOREM 4.1. *(Gysin) Let $\pi : S(E) \to M$ be an oriented sphere bundle with fiber S^k. Then there is a long exact sequence*

$$\cdots \to H^n(S(E)) \xrightarrow{\pi_*} H^{n-k}(M) \xrightarrow{\wedge e} H^{n+1}(M)$$
$$\xrightarrow{\pi^*} H^{n+1}(S(E)) \to \cdots,$$

in which the maps $\pi_*, \wedge e,$ and π^* are integration along the fiber, multiplication by the Euler class, and the nature pullback, respectively.

THEREOM 4.2. *Let X be a compact manifold and let $f : X \to Y$ be transversal to a submanifold $Z \subset Y$, then the*

premage $f^{-1}(Z)$ is a submanifold of X. Moreover, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

The most important and readily visualized special situation concerns the transversality of the inclusion map i of one submanifold $X \subset Y$ with another submanifold $Z \subset Y$. To say a point $x \in X$ belongs to the preimage $i^{-1}(Z)$ simply means that x belongs to the intersection $X \cap Z$. Also, the derivative $di_x : T_x(X) \to T_x(Y)$ is merely the inclusion map of $T_x(X)$ into $T_x(Y)$. So $i \pitchfork Z$ if and only if, for every $x \in X \cap Z$,

$$T_x(X) + T_x(Z) = T_x(Y)$$

THEOREM 4.3. *Let $f : X \to Y$ be a map transversal to a submanifold Z in Y and let $W = f^{-1}(Z)$. Then $T_x(W)$ is the premage of $T_{f(x)}(Z)$ under the linear map $df_x : T_x(X) \to T_{f(x)}(Y)$.*

Let $N_x(W; X)$ be the orthogonal complement to $T_x W$ in $T_x X$, i.e., $N_x(W; X) \oplus T_x W = T_x X$. Because of $df_x(T_x) + T_z Z = T_z Y = df_x(N_x(W; X)) + T_z Z$, for each $f(x) = z \in Z$, the orientations of Z and Y induces an orientations $df_x(T(W; X))$. By the isomorphism df_x, the orientation on $df_x(N_x(W; X))$ defines an orientations on $N_x(W; X)$. Finally the orientation on $N_x(W; X)$ and $T_x X$ define orientations on each tangent space $T_x(W)$.

In particular if two subspaces X_1 and X_2 of X are transversal and have complementary dimensions, then each point of $X_1 \cap X_2$ has ± 1-orientation which is defined by the inclusion map $i : X_1 \to X_2$.

Suppose that a function $f : M \to R$ is a Morse function satisfying the Morse-Smale type. For any two critical points x and y of f, $\mathcal{M}(x, y) \cong W^u(x) \cap W^s(y)$ is a smooth submanifold of M with dimension $ind(x) - ind(y)$. Let $s : M \to E$ be a generic section. The preimage $W = s^{-1}(0)$ of the zero section has codimension n. By genericity, $W \cap \mathcal{M}(x, y) = \emptyset$ if $ind(x) -$

$ind(y) < n$, $W \cap \mathcal{M}(x,y)$ is a set of finite points if $ind(x) - ind(y) = n$.

Let $C^{\iota} = \bigoplus_{deg\rho^{\iota}=i} \mathbb{Z}\rho^{\iota}$ and the exterior differentiation $d : C^{\iota} \to C^{\iota+1}$ is defined by connecting homomorphisms. We want to define a chain map $c : C^{\iota} \to C^{\iota+n}$ by

$$c(\rho^{\iota}) = \sum \sharp(W \cap \mathcal{M}(\rho^{\iota}, \rho^{\iota+n})\rho^{\iota+n}$$
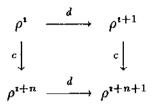
, where the number $\sharp$ is the sign.

THEOREM 4.4. *The linear map* $c : C^{\iota} \to C^{\iota+n}$ *is a chain map from* C^* *into itself.*

Proof. Consider the following diagram

$$\cdots \to C^{\iota} \xrightarrow{d} C^{\iota+1} \xrightarrow{d} C^{\iota+2} \to \cdots$$
$$\quad c\downarrow \qquad\qquad c\downarrow \qquad\qquad c\downarrow$$
$$\cdots \to C^{i+n} \xrightarrow{d} C^{\iota+n+1} \xrightarrow{d} C^{\iota+n+2} \to \cdots$$

For each generator ρ^{ι} in C^{ι} since $c(\rho^{\iota}) = \sum_{\rho^{\iota+n} \in C^{\iota+n}} \sharp(W \cap \mathcal{M}(\rho^{\iota}, \rho^{\iota+n}))\rho^{\iota+n}$,

$d(c(\rho^{\iota}))$

$$= \sum_{\rho^{\iota+n} \in C^{\iota+n}} \sharp(W \cap \mathcal{M}(\rho^{\iota}, \rho^{\iota+n}))d(\rho^{\iota+n})$$

$$= \sum_{\rho^{\iota+n} \in C^{\iota+n}} \sharp(W \cap \mathcal{M}(\rho^{\iota}, \rho^{\iota+n}))$$
$$\sum_{\rho^{\iota+n+1} \in C^{\iota+n+1}} n(\rho^{\iota+n}, \rho^{\iota+n+1})\rho^{\iota+n+1}$$

$$= \sum_{\rho^{\iota+n+1} \in C^{\iota+n+1}} \left(\sum_{\rho^{\iota+n} \in C^{\iota+n}} \sharp(W \cap \mathcal{M}(\rho^{\iota}, \rho^{\iota+n})) \right.$$
$$n(\rho^{\iota+n}, \rho^{\iota+n+1})\rho^{\iota+n+1},$$

$c(d(\rho^{\iota}))$

$$= \sum_{\rho^{\iota+n+1} \in C^{\iota+n+1}} \left(\sum_{\rho^{\iota+1} \in C^{\iota+1}} n(\rho^{\iota}, \rho^{\iota+1}) \right.$$
$$\sharp(W \cap \mathcal{M}(\rho^{\iota+1}, \rho^{\iota+n+1}))\rho^{\iota+n+1}$$

The gradient flow lines and the Euler class determine the following commutative digram up to sign

$$
\begin{array}{ccc}
\rho^{\imath} & \xrightarrow{\ d\ } & \rho^{\imath+1} \\
{\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow \\
\rho^{\imath+n} & \xrightarrow{\ d\ } & \rho^{\imath+n+1}
\end{array}
$$

Thus

$$
\sum_{\rho^{\imath+n}\in C^{\imath+n}} \#(W\cap \mathcal{M}(\rho^{\imath},\rho^{\imath+n}))n(\rho^{\imath+n},\rho^{\imath+n+1})
$$

$$
= \sum_{\rho^{\imath+1}\in C^{\imath+1}} n(\rho^{\imath},\rho^{\imath+1})\#(W\cap \mathcal{M}(\rho^{\imath+1},\rho^{\imath+n+1}))
$$

Thus $c(d\rho^{\imath}) = d(c\rho^{\imath})$.

We define a new coboundary map $\bar{d}: C^{*}\oplus C^{*}\to C^{*}\oplus C^{*}$ by

$$
\bar{d} = \begin{pmatrix} d & c \\ 0 & d \end{pmatrix} : C^{\imath}\oplus C^{\imath-n+1}\to C^{\imath+1}\oplus C^{\imath-n+2}
$$

THEOREM 4.5. *(1) $\bar{d}^2 = 0$ if and only if c is a chain map i.e,. $dc + cd = 0$ (2) The cohomology group for $(C^{*}\oplus C^{*},\bar{d})$ is isomorphic to the cohomology group of the sphere bundle. i.e,. $H^p(C^{*}\oplus C^{*},\bar{d}) \simeq H^p(S(E))$.*

Proof. (1) Since

$$
\bar{d}^2 = \begin{pmatrix} d & c \\ 0 & d \end{pmatrix} : C^{\imath}\oplus C^{\imath-n+1}\to C^{\imath+1}\oplus C^{\imath-n+2},
$$

$$\bar{d}^2 = \begin{pmatrix} d & c \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} d & c \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} d^2 & dc + cd \\ 0 & d^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & dc + cd \\ 0 & 0 \end{pmatrix}$$

$$= 0$$

Hence $dc + cd = 0$.

(2) To find the cohomology group for $(C^* \oplus C^*, \bar{d})$ at p, we consider the chain complex

$$\rightarrow C^{p-1} \oplus C^{p-n} \xrightarrow{\bar{d}_{p-1}} C^p \oplus C^{p-n+1} \xrightarrow{\bar{d}_p} C^{p+1} \oplus C^{p-n+2} \rightarrow$$

The p-th comology group $H^p(C^* \oplus C^*, \bar{d}) = Ker \bar{d}_p / Im \bar{d}_{p-1} =$

$$\frac{\{(\rho^p, \rho^{p-n+1}) \in C^p \oplus C^{p-n+1} | d_p(\rho^p) + c(\rho^{p-n+1}) = 0 = \bar{d}_{p-n+1}(\rho^{p-n+1})\}}{\{(d_{p-1}(\rho^{p-1}) + c(\rho^{p-n}), d_{p-n}(\rho^{p-n})) | (\rho^{p-1}, \rho^{p-n}) \in C^{p-1} \oplus C^{p-n}\}}$$

For each $(\rho^p, \rho^{p-n+1}) \in H^p(C^* \oplus C^*, \bar{d})$, $\bar{d}_{p-n+1}(\rho^{p-n+1}) = 0$. Let $e \in H^{n-1}(S^{n-1})$ be a generator. We may consider $e \in H^{n-1}(S^{n-1}) \simeq H^n(R^n, R^n - \{0\})$ as an orientation on each fiber of the vector bundle$E$. We define a homomorphism,

$$\phi : H^p(C^* \oplus C^*, \bar{d}) \rightarrow H^p(S(E))$$

by

$$\phi(\rho^p, \rho^{p-n+1}) = \pi^*(\rho^p) + \pi^*(\rho^{p-n+1}) \wedge e.$$

Then ϕ is an isomorphism.

Let $f : M \rightarrow R$ be a Morse function which satisfies the Morse-Smale condition. We would like to construct a Morse-Smale function $F : S(E) \rightarrow R$ on the total space S(E) of the sphere bundle and investigate the topology on it. Suppose

$s : M \to E$ is a generic section, namely transversal to zero section of the vector bundle E. The preimage $s^{-1}(0)$ of the zero section is a submanifold of M with codimension n. We define a function

$$g : S(E) \to R$$

by $g(v_x) = <v_x, s\pi(v_x)>$ for each $v_x \in S(E)$ where the inner product $<,>$ is defined on E. Thus we define a function $F : S(E) \to R$ via the function f and g,

$$F(v_x) = \pi^* f(v_x) + g(v_x).$$

THEOREM 4.6. *[10] (1) $F : S(E) \to R$ is a smooth Morse function. (2) The complex $C(S(E), F)$ on the total space $S(E)$ defined by F is isomorphic to the complex $(C(X, f) \oplus C(X, f), \bar{d})$ defined by f and d(defined by a chain map c).*

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