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ON TIME DECAYS FOR SOME SEMILINEAR WAVE EQUATION WITH A DISSIPATIVE TERM

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0. Introduction

In this paper we investigate on the decay properties of the solution and its derivatives to the Cauchy problem for the semilinear wave equation with the dissipative term :

$$(0.1)\qquad \begin{cases} u_{tt} - \Delta u + u_t + f(u) = 0 \quad \text{in} \quad R^N \times [0, \infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x)\,, \end{cases}$$

where $f(u)$ is a nonlinear function like $f(u) = |u|^\alpha u$, $\alpha > 0$ (i.e., $f(u)u \geq 0$).

In the case $0 < \alpha < 4/[N-2]^+$, we have already studied on certain decay rates for the solution and its derivatives of the equation (0.1). (See Kawashima, Nakao & Ono [10].) For example, in the typical case $N = 3$ and $\alpha = 2$,

$$\|D_x^k D_t^l u(t)\| \leq C_{[m+1]}(1+t)^{-\frac{k+l}{2}-\eta}$$

for $0 \leq k, l, k+l \leq m+1$, $m \geq 1$, where η is some positive number and $C_{[m+1]}$ is some positive constant given by (0.5b) and (0.6) respectively. Here, we note that this results do not need a smallness condition to initial data $(u_0, u_1) \in H^{m+1} \cap L^r \times H^m \cap L^r$, $m \geq 1$, $1 \leq r \leq 2$. Moreover, this results are effective even for weak solutions.

The purpose of this paper is to improve previous results in [10] for its derivatives with respect to t of the solution of the equation (0.1). To this end, we need to improve L^p-L^q-estimates of the linear dissipative wave equation which is (0.1) with $f(u) \equiv f(x,t)$. So, we shall give more detailed L^p-L^q-estimates than [10] do. (For the nondissipative case, see Brenner [1], Pecher [18], Mochizuki [13], Ginibre & Velo [5], etc.) Under initial data (u_0, u_1) is sufficiently small, similar results on decay

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property of classical solutions are given by Matsumura [11]. Recently, Rack [19] have given some L^p-L^q-estimates for the dissipative equation in more general setting for the case $N = 3$ and $f(u) \equiv 0$. Moreover, in this paper, adding the improved L^p-L^q-estimates, we utilize the L^2-norm of the equation (0.1) and its differentiated equations, e.g., $\|D_t^i u_t\| \le \|D_t^i(u_{tt} - \Delta u + f(u))\|$, $i = 0, 1, \cdots$, etc. Then, we regard a decay rate of $\|u_t(t)\|$ as a decay rate $\|\Delta u(t)\|$ in a certain case.

The existence and uniqueness for a solution of usual nondissipative wave equations (e.g., u_t be removed from (0.1))have already been proved by many scientists. (Cf. Jörgens [9], Strauss [22], Pecher [18], Brenner & W. v. Wahl [3], Grillakis [6], [7], Struwe [23], etc.) Thus, we can obtain the similar results for our equation (0.1) by trivial modifications.

Recently, we have give some results the case when we do not need an assumption such that $f(u)u \ge 0$ (for example $f(u) = -|u|^\alpha u$) under some smallness condition of (u_0, u_1), or when the nonlinear dissipative term case (see [15], [16]). Moreover, we have derived precise results on the decay of solutions for the equation (0.1) with $f(u)$ replaced by a nonlinear function $g(u_t)$ like $g(u_t) = |u_t|^\beta u_t$, $\beta > 0$ (see [17]).

The content of this paper is as follows. In Section 1 we state some known Lemmas (but we omit their proof). In Section 2 we give energy decay estimates and L^p-L^q-estimates for a linear dissipative wave equation. In Section 3 we state our hypotheses on the nonlinear term $f(u)$ and our main results for the nonlinear dissipative wave equation (0.1). Their proof consist of several steps and will be given in Section $3 \sim 6$.

"*Notation*" We shall denote by D_x^k, $k \ge 0$ integer, any partial differential operators of order k with respect to the space variables x_i, $i = 1, 2, \cdots, N$. The differentiation of order l with respect to the time t is denoted by D_t^l or $\left(\frac{d}{dt}\right)^l$. In particular, D denotes a partial differential operator D_x or D_t. We use only standard function spaces H_p^s ($L^p \equiv H_p^0$, $H^s \equiv H_2^s$) equipped with the norm :

$$\|u\|_{H_p^s} \equiv \|\mathcal{F}^{-1}\{< \xi >^s \widehat{u}(\xi)\}\|_p, \tag{0.2}$$

where $< \xi >= \sqrt{1 + |\xi|^2}$ and $\|\cdot\|_p$ denotes the usual L^p-norm (we use

$\|\cdot\|$ for $\|\cdot\|_2$), and $\mathcal{F}$ denotes the Fourier transform :

$$(0.3) \qquad \mathcal{F}\{u(x)\}(\xi) \equiv \widehat{u}(\xi) \equiv \frac{1}{\sqrt{2\pi}^N}\int e^{-i<\xi,x>}u(x)\,dx\,.$$

We denote special notations by

$$(0.4) \qquad \omega_{k,l} \equiv \frac{k}{2}+l \qquad \text{for} \quad k,l=0,1,2,\cdots\,,$$

$$(0.5a) \qquad \eta_r^* \equiv \frac{N}{2}(\frac{1}{r}-\frac{1}{2}) \quad \text{for} \quad 1 \le r \le 2\,,$$

and

$$(0.5b) \qquad \eta \equiv \begin{cases} \min\{\eta_r^*\,,\,N\alpha/4\} & \text{if} \quad \alpha > 4/N\,, \\ 0 & \text{if} \quad \alpha \le 4/N\,. \end{cases}$$

Here, we note that

$$(0.5c) \qquad \eta = \min\{\eta_r^*\,,\,N\alpha/4\} = \eta_r^* \quad \text{if} \quad N \le 4 \text{ and } \alpha > 4/N\,.$$

And, we denote by C various positive constants independent of (u_0, u_1) and , in particular, denote by $C_{[m+1]}$ various positive constant depending on $\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}$ or $\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \|u_0\|_r + \|u_1\|_r$ and other known constants, namely,

$$(0.6) \qquad C_{[m+1]} \equiv c(\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \|u_0\|_r + \|u_1\|_r)\,.$$

Moreover, let $[a]^+ = \max\{0, a\}$, where $1/[a]^+ = \infty$ if $[a]^+ = 0$, and let pairs of conjugate indices be written as p, p', where $1/p + 1/p' = 1$.

A. Preliminaries and Linear Dissipative Wave Equation

1. Some Lemmas

We use the following Lemmas through this paper (but we omit their proof).

The first one is well known.

LEMMA 1.1. (Gagliardo-Nirenberg) *Let $1 \leq r < p \leq \infty$, $1 \leq q \leq p$ and $0 \leq k \leq m$. Then, the inequality*

$$\|D_x^k v\|_p \leq C_0 \|D_x^m v\|_q^\theta \|v\|_r^{1-\theta} \quad \text{for} \quad v \in H_q^m \cap L^r \tag{1.1a}$$

holds with some constant $C_0 > 0$ and

$$\theta = \Big(\frac{1}{p} - \frac{k}{N} - \frac{1}{r}\Big)\Big(\frac{1}{q} - \frac{m}{N} - \frac{1}{r}\Big)^{-1}, \tag{1.1b}$$

provided that $0 < \theta \leq 1$ ($0 < \theta < 1$ if $1 < q < \infty$ and $m - N/q$ is a nonnegative integer).

The second one is powerful in deriving decay rates of energies.

LEMMA 1.2. (Nakao) *Let $\phi(t)$ be a nonnegative function on $R^+ = [0, \infty)$, satisfying*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\alpha} \leq k_0 (1+t)^\beta \{\phi(t) - \phi(t+1)\} + h(t) \tag{1.2a}$$

for some $k_0 > 0, \alpha > 0, \beta < 1$, and a function $h(t)$ with

$$0 \leq h(t) \leq k_1 (1+t)^{-\gamma} \tag{1.2b}$$

for some $k_1 > 0$ and $\gamma > 0$. Then, $\phi(t)$ has a decay property

$$\phi(t) \leq C_0 (1+t)^{-\theta}, \qquad \theta = \min\Big\{\frac{1-\beta}{\alpha}, \frac{\gamma}{1+\alpha}\Big\}, \tag{1.2c}$$

where C_0 denotes a positive constant depending on $\phi(0)$ and other known constants.

REMARK 1.1. The proof of this lemma is given in Nakao [14] under a little strong assumption $h(t) = o(t^{-\gamma})$ with $\gamma = (1+\alpha)(1-\beta)/\alpha$ as $t \to \infty$. The perfect proof of this lemma is given in [10].

The third one is useful in deriving L^q-estimates.

LEMMA 1.3. ([10]) *Let $y(t)$ be a nonnegative function on $[0, T), T > 0$ (possibly $T = \infty$), and satisfy the integral inequality*

$$y(t) \leq k_0 (1+t)^{-\alpha} + k_1 \int_0^t (1+t-s)^{-\beta} (1+s)^{-\gamma} y(s)^\mu \, ds \tag{1.3a}$$

for some $k_0, k_1 > 0$, $\alpha, \beta, \gamma \geq 0$ *and* $0 \leq \mu < 1$. *Then, the function* $y(t)$ *has*

$$y(t) \leq C_0(1+t)^{-\theta} \tag{1.3b}$$

for some constant $C_0 > 0$ and

$$\theta = \min\left\{\alpha, \beta, \frac{\gamma}{1-\mu}, \frac{\beta+\gamma-1}{1-\mu}\right\}, \tag{1.3c}$$

with the following exceptional case : If $\alpha \geq \widetilde{\theta}$ *and* $(\beta+\gamma-1)/(1-\mu) = \widetilde{\theta} \leq 1$, *where*

$$\widetilde{\theta} = \min\left\{\beta, \frac{\gamma}{1-\mu}\right\}, \tag{1.3d}$$

then, the function $y(t)$ *has*

$$y(t) \leq C_0(1+t)^{-\widetilde{\theta}}\Big(\log(2+t)\Big)^{1/(1-\mu)}. \tag{1.3e}$$

REMARK 1.2. Once we know $y(t)$ is a bounded function, we can apply Lemma 1.3 also to the case $\mu \geq 1$. In particular, if $\gamma > 0$ and $\beta + \gamma - 1 > 0$, we obtain (1.3b) with

$$\theta = \min\{\alpha\,,\,\beta\} \quad \left(= \min\left\{\alpha, \beta, \frac{\gamma}{[1-\mu]^+}, \frac{\beta+\gamma-1}{[1-\mu]^+}\right\}\right). \tag{1.3f}$$

Moreover, we note that even for the exceptional case, (1.3b) is valid if θ is replaced by $\bar{\theta} \equiv \theta - \varepsilon$, $0 < \varepsilon \ll 1$.

The forth one is well known.

LEMMA 1.4. (Fourier Multiplier (cf. [8], [12], [21]) *Let* $f(\xi)$ *be class* C^m, $m > N/2$, *in the complement of the origin of* R^N_ξ, *satisfying*

$$|\xi|^k |D^k_\xi f(\xi)| \leq M_0\,, \quad 0 \leq k \leq m\,, \tag{1.4a}$$

with some $M_0 > 0$. *Then, for* $1 < p < \infty$ *and* $v \in C^\infty_o(R^N)$,

$$\|\mathcal{F}^{-1}\{f(\xi)\widehat{v}(\xi)\}\|_p \leq C_0 M_0 \|v\|_p\,, \tag{1.4b}$$

where C_0 is a certain positive constant independent of M_0.

2. Energy Decay and L^p-L^q-estimates for Linear Dissipative Wave Equation

In this section we consider the linear wave equation with a dissipative term :

$$(2.1) \qquad \begin{cases} u_{tt} - \Delta u + u_t = f(x,t) & \text{in} \quad R^N \times [0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x)\,. \end{cases}$$

First, we give here a decay property for the energy $E(t) \equiv \|D_x u(t)\|^2 + \|D_t u(t)\|^2$ to the linear equation (2.1), which is useful in deriving the decay rates of its derivatives to the nonlinear equation (0.1). The following result is proved by Propositions 4.1 and 4.2 in [10]. (We omit here the proof.)

LEMMA 2.1. *Let $(u_0, u_1) \in H^1 \times L^2$ and let $f(t) \in L^2_{loc}([0,\infty); L^2)$. Suppose that $u(t)$ is a solution of the linear equation (2.1) which belongs to $C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$. Further, we assume that*

$$(2.2a) \qquad \|u(t)\|^2 \le k_0(1+t)^{-2a}$$

and for $n = 1, 2, \cdots$,

$$(2.2b) \qquad \|f(t)\|^2 \le \sum_{j=1}^{n} k_j(1+t)^{-2b_j} E(t)^{\mu_j}$$

with some $k_0, k_j > 0, a \ge 0, b_j > 0$, and $0 < \mu_j < 1$ for $j = 1, 2, \cdots, n$. Then, the energy $E(t)$ has the decay property

$$(2.2c) \qquad E(t) \equiv \|D_x u(t)\|^2 + \|D_t u(t)\|^2 \le C_1(1+t)^{-2\theta}\,,$$

where C_1 is a positive constant depending on $E(0)$ and other known constants, and $\theta > 0$ is given by

$$(2.2d) \qquad \theta = \min\left\{\frac{1}{2} + a\,,\ \min_{1\le j\le n}\left\{\frac{b_j}{[1-\mu_j]^+}\,,\ \frac{a+b_j}{2-\mu_j}\right\}\right\}.$$

Next, we shall derive L^p-L^q-estimates for the solution and its derivatives of the linear equation (2.1). Now, we see that the solution $u(t)$ is given through Fourier transform :

$$\widehat{u}(\xi,t) = \widehat{u_L}(\xi,t) + \widehat{I_f}(\xi,t). \tag{2.3a}$$

Here, we define

(2.3b)

$$\widehat{u_L}(\xi,t) = \frac{1}{2}\big(\phi_1(\xi,t) + \phi_2(\xi,t)\big)\widehat{u_0}(\xi) + \phi_2(\xi,t)\widehat{u_1}(\xi),$$

and

(2.3c)

$$\widehat{I_f}(\xi,t) = \int_0^t \phi_2(\xi,t-s)\widehat{f}(\xi,s)\,ds,$$

where we set

$$\begin{aligned}\phi_1(\xi,t) &= e^{-t/2}\{e^{\lambda t/2} + e^{-\lambda t/2}\}\\ &= \begin{cases} 2e^{-t/2}\cosh(\lambda t/2) & \text{if} \quad |\xi| \le 1/2,\\ 2e^{-t/2}\cos(\lambda_* t/2) & \text{if} \quad |\xi| > 1/2,\end{cases}\end{aligned} \tag{2.3d}$$

and

$$\begin{aligned}\phi_2(\xi,t) &= \frac{1}{\lambda}e^{-t/2}\{e^{\lambda t/2} - e^{-\lambda t/2}\}\\ &= \begin{cases} 2\lambda^{-1}e^{-t/2}\sinh(\lambda t/2) & \text{if} \quad |\xi| \le 1/2,\\ 2\lambda^{-1}e^{-t/2}\sin(\lambda_* t/2) & \text{if} \quad |\xi| > 1/2\end{cases}\end{aligned} \tag{2.3e}$$

with $\lambda = \sqrt{1-4|\xi|^2}$ and $\lambda_* = \sqrt{4|\xi|^2-1}$ $(= \sqrt{-1}\,\lambda)$. Then, we see easy that for $l = 0,1,2,\cdots$,

(2.4a)

$$\Big(\frac{d}{dt}\Big)^l\phi_i(\xi,t) = \begin{cases} \Big(\dfrac{-1+\lambda}{2}\Big)^l e^{(-1+\lambda)t/2} + \Big(\dfrac{-1-\lambda}{2}\Big)^l e^{(-1-\lambda)t/2}, & i = 1,\\ \dfrac{1}{\lambda}\Big[\Big(\dfrac{-1+\lambda}{2}\Big)^l e^{(-1+\lambda)t/2} - \Big(\dfrac{-1-\lambda}{2}\Big)^l e^{(-1-\lambda)t/2}\Big], & i = 2.\end{cases}$$

In particular, if $|\xi| > 1/2$ we see

(2.4b)

$$\left(\frac{d}{dt}\right)^l \phi_i(\xi,t) = \begin{cases} 2^{l-1}e^{-t/2}\sum_{j=0}^{l}(-1)^j\lambda_*^{l-j}\cos\Big(\dfrac{\lambda_* t+(l-j)\pi}{2}\Big), & i=1, \\ 2^{l-1}e^{-t/2}\sum_{j=0}^{l}(-1)^j\lambda_*^{l-j-1}\sin\Big(\dfrac{\lambda_* t+(l-j)\pi}{2}\Big), & i=2. \end{cases}$$

Now, we take $\chi_j \in C^\infty(R^N)$, $j = 1,2,3$, such that

(2.5a)

$$\chi_1(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1/4, \\ 0 & \text{if} \quad |\xi| \ge 1/3, \end{cases}$$

(2.5b)

$$\chi_3(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| \le 2/3, \\ 1 & \text{if} \quad |\xi| \ge 3/4, \end{cases}$$

and

(2.5c)

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi), \quad \text{i.e.,} \quad \sum_{j=1}^{3}\chi_j(\xi) = 1.$$

The purpose here is to improve the L^p-L^q-estimates for the solution and its derivatives of the linear dissipative wave equation (2.1) given in a previous paper [10] (with the case $k = l = 0$). To this we give the L^p-L^q-estimates for a function v which belongs to $C_o^\infty(R^N)$ and is divided up the function $\chi_j(\xi)$, $j = 1,2,3$, by the following forms.

THEOREM A1. *Let v belong to $C_o^\infty(R^N)$,and let k and l be nonnegative integer. Then, we have :*

(i) *For $1 \le p \le 2$, $2 \le q \le \infty$, $0 \le j \le k$, and $\gamma \in R$,*

$$J_1(t) \equiv \|\mathcal{F}^{-1}\{\chi_1(\xi)(\sqrt{-1}\,\xi)^k\left(\frac{d}{dt}\right)^l\phi_i(\xi,t)\hat{v}(\xi)\}\|_{H^q_\gamma}$$

$$\le C(1+t)^{-(\frac{k-j}{2}+l+\frac{N}{2}(\frac{1}{p}-\frac{1}{q}))}\|D_x^j v\|_p, \quad i = 1,2. \tag{2.6}$$

(ii) *For* $1 \le p \le 2$, $0 \le j \le k$, *and* $\gamma \in R$,

$$J_2(t) \equiv \|\mathcal{F}^{-1}\{\chi_2(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\phi_i(\xi,t)\widehat{v}(\xi)\}\|_{H^\gamma_{p'}}$$

$$\le C\,e^{-\nu t}\|D_x^j v\|_p\,,\quad i=1,2\,, \tag{2.7}$$

with some $0<\nu<1/2$.

(iii) *For* $1 \le p \le 2$, $\gamma \in R$, *and an integer* $m \ge 0$,

$$J_3(t) \equiv \|\mathcal{F}^{-1}\{\chi_3(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\phi_i(\xi,t)\widehat{v}(\xi)\}\|_{H^\gamma_{p'}}$$

$$\le C\,e^{-\nu t}|t|^{-(N-1)(\frac{1}{p}-\frac{1}{2})}\|D_x^m v\|_p\,,\quad i=1,2\,, \tag{2.8a}$$

with some $0<\nu<1/2$, *provided that*

$$k+l+\gamma+(N+1)\big(\frac{1}{p}-\frac{1}{2}\big) \le \begin{cases} m & \textit{if}\quad i=1\,,\\ m+1 & \text{if}\quad i=2\,. \end{cases} \tag{2.8b}$$

(iv) *For* $\gamma \in R$,

$$J_4(t) \equiv \|\mathcal{F}^{-1}\{\chi_3(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\phi_i(\xi,t)\widehat{v}(\xi)\}\|_{H^\gamma}$$

$$\le C\,e^{-\nu t}\|v\|_{H^s}\,,\quad i=1,2\,, \tag{2.9a}$$

with some $0<\nu<1/2$ *and*

$$s = \begin{cases} k+l+\gamma & \textit{if}\quad i=1\,,\\ k+l+\gamma-1 & \text{if}\quad i=2\,. \end{cases} \tag{2.9b}$$

Proof. We consider the case $i=2$ only (because that the case $i=1$ is proved in the same way). First, we shall show (2.6). Noting the (2.4a), we have from the Hausdorff-Young inequality that

$$\begin{aligned} J_1(t) &\le \|\chi_1(\xi) <\xi>^\gamma(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\phi_2(\xi,t)\widehat{v}(\xi)\|_{q'}\\ &\le \|\chi_1(\xi) <\xi>^\gamma(\sqrt{-1}\,\xi)^k\frac{1}{\lambda}\Big(\frac{-1+\lambda}{2}\Big)^l e^{(-1+\lambda)t/2}\widehat{v}(\xi)\|_{q'}\\ &\quad + \|\chi_1(\xi) <\xi>^\gamma(\sqrt{-1}\,\xi)^k\frac{1}{\lambda}\Big(\frac{-1-\lambda}{2}\Big)^l e^{(-1-\lambda)t/2}\widehat{v}(\xi)\|_{q'}\,, \end{aligned}$$

where $\lambda=\sqrt{1-4|\xi|^2}$. Here, since $|\xi|\le 1/3$ (see (2.5a)), we see that $-4|\xi|^2\le -1+\lambda\le -2|\xi|^2$ and $\sqrt{5}/3\le\lambda\le 1$. Thus, we have that for any $0\le j\le k$,

$$J_1(t)\le C\,\|\chi_1(\xi)|\xi|^{k-j}|\xi|^{2l}e^{-|\xi|^2t}\widehat{D_x^j v}(\xi)\|_{q'}$$

$$+\,C\,e^{-t/2}\|\chi_1(\xi)\widehat{D_x^j v}(\xi)\|_{q'}\equiv J_1^{(1)}(t)+J_1^{(2)}(t)\,. \tag{2.10}$$

Here, we see from the Hölder inequality that

$$J_1^{(1)}(t)\le C\left(\int_{|\xi|\le 1/3}|\xi|^{p'q'(k-j+2l)/(p'-q')}e^{-p'q'|\xi|^2t/(p'-q')}d\xi\right)^{(p'-q')/p'q'}$$

$$\|\widehat{D_x^j v}\|_{p'}\,,$$

and we see from the Hausdorff-Young inequality that

$$J_1^{(1)}(t)\le C\left(\int_0^{1/3}|\xi|^{p'q'(k-j+2l)/(p'-q')+N-1}\right.$$

$$\left.e^{-p'q'|\xi|^2t/(p'-q')}d|\xi|\right)^{(p'-q')/p'q'}\|D_x^j v\|_p$$

$$\le C\,(1+t)^{-(\frac{k-j}{2}+l+\frac{N}{2}(\frac{1}{p}-\frac{1}{q}))}\|D_x^j v\|_{p'}\,, \tag{2.11}$$

using the following inequality : If $0<\varepsilon<1$, $k_1\ge 0$, and $k_2>0$, then

$$\int_0^{\varepsilon}|\xi|^{k_1}e^{-k_2|\xi|^2t}d|\xi|\le C\,(1+t)^{-(k_1+1)/2}\,. \tag{2.12}$$

On the other hand, we see easy

$$J_1^{(2)}(t)\le C\,e^{-t/2}\|\widehat{D_x^j v}\|_{p'}\le C\,e^{-t/2}\|D_x^j v\|_p\,. \tag{2.13}$$

Thus, from the above estimates (2.10), (2.11), and (2.13), we obtain the desired estimate (2.6).

Next, we shall show (2.7). Noting that $supp\,\chi_2(\xi)\subset\{1/4\le|\xi|\le 3/4\}$ (see (2.5c)), we see that for any j such that $0\le j\le k$,

$$\sup_{1/4\le|\xi|\le 3/4}|<\xi>^{\gamma}(\sqrt{-1}\xi)^{k-j}(\frac{d}{dt})^l\phi_2(\xi,t)|$$

$$\le C\,|t|\,e^{-t/2}\sup_{1/4\le|\xi|\le 3/4}\left|\frac{e^{\lambda t/2}-e^{-\lambda t/2}}{\lambda t/2}\right|\le C\,e^{-\nu t} \tag{2.14}$$

with some $0 < \nu < 1/2$. Thus, applying the Hausdorff-Young inequality, we arrive at the desired estimate (2.7).

Moreover, we shall prove (2.8). Noting (2.4b), we see

$$J_3(t) = 2^{l-1} e^{-t/2} \|\mathcal{F}^{-1}\{\chi_3(\xi) < \xi >^\gamma (\sqrt{-1}\,\xi)^k \sum_{j=0}^{l} (-1)^j \lambda_*^{l-j-1} \sin\Big(\frac{\lambda_* t + (l-j)\pi}{2}\Big)\}\|_{p'} .$$

Here, since $supp\,\chi_3(\xi) \subset \{|\xi| \geq 2/3\}$ (see (2.5b)) and

$$\sup_{|\xi|\geq 2/3} \sum_{n=0}^{N} \Big| D_\xi^n \Big\{ \frac{< \xi >^\gamma (\sqrt{-1}\,\xi)^k \sum_{j=0}^{l} (-1)^j \lambda_*^{l-j-1} |\xi|^{(N+1)(\frac{1}{p}-\frac{1}{2})} \sin\Big(\frac{\lambda_* t + (l-j)\pi}{2}\Big)}{|\xi|^m \sin|\xi| t} \Big\} \Big|$$
$$\leq C\,(1+t)^N ,$$

where $\gamma + k + (l - j - 1) + (N+1)(1/p - 1/2) \leq m$ for any j such that $0 \leq j \leq l$ (by (2.8b)), we can choose

$$\frac{< \xi >^\gamma (\sqrt{-1}\,\xi)^k \sum_{j=0}^{l} (-1)^j \lambda_*^{l-j-1} |\xi|^{(N+1)(\frac{1}{p}-\frac{1}{2})} \sin\Big(\frac{\lambda_* t + (l-j)\pi}{2}\Big)}{|\xi|^m \sin|\xi| t}$$

as a Fourier multiplier on $supp\,\chi_3(\xi)$ (see Lemma 1.4) to get

$$J_3(t) \leq C\,(1+t)^N e^{-t/2} \|\mathcal{F}^{-1}\Big\{\chi_3(\xi) \frac{\sin|\xi|t}{|\xi|^{(N+1)(\frac{1}{p}-\frac{1}{2})}} |\xi|^m \widehat{v}(\xi)\Big\}\|_{p'} .$$

Now, we take $\chi \in C^\infty(R)$ such that

$$\chi(s) = \begin{cases} 0 & \text{if} \quad |s| \leq 1/4 , \\ 1 & \text{if} \quad |s| \geq 1/2 . \end{cases}$$

Then, we can choose

$$|\xi|^m \Big/ \sum_{n=0}^{N} \chi(\xi_n) |\xi_n|^m$$

as a Fourier multiplier on $\operatorname{supp}\chi_3(\xi)$ to get

$$J_3(t) \le C\,(1+t)^N e^{-t/2} \sum_{n=0}^{N} \|\mathcal{F}^{-1}\Big\{\chi_3(\xi)\frac{\sin|\xi|t}{|\xi|^{(N+1)(\frac{1}{p}-\frac{1}{2})}}\chi(\xi_n)|\xi_n|^m\widehat{v}(\xi)\Big\}\|_{p'}\,.$$

Furthermore, we take

$$\chi(\xi_n)|\xi|^m\Big/(\sqrt{-1}\xi_n)^m$$

as a Fourier multiplier. Then, we have

$$J_3(t) \le C\,(1+t)^N e^{-t/2} \sum_{n=0}^{N} \|\mathcal{F}^{-1}\Big\{\chi_3(\xi)\frac{\sin|\xi|t}{|\xi|^{(N+1)(\frac{1}{p}-\frac{1}{2})}}\widehat{D^m_{x_n}v}(\xi)\Big\}\|_{p'}\,.$$

Finally, using an estimate used for the L^p-$L^{p'}$-estimate of the usual wave equation (cf. [1], [18], [20]), we obtain

$$\begin{aligned} J_3(t) &\le C\,(1+t)^N e^{-t/2}|t|^{-2N(\frac{1}{p}-\frac{1}{2})+(N+1)(\frac{1}{p}-\frac{1}{2})} \sum_{n=0}^{N}\|D^m_{x_n}v\|_p \\ &\le C\,e^{-\nu t}|t|^{-(N-1)(\frac{1}{p}-\frac{1}{2})}\|D^m_x v\|_p \end{aligned} \tag{2.15}$$

with some $0<\nu<1/2$, where $k+l+\gamma+(N+1)(1/p-1/2)\le m+1$, which implies the desired estimate (2.8).

Finally, we note that (2.9) follows easily from the Plancherel theorem. □

Summing up the above estimates (2.6), (2.7), and (2.8) in Theorem A1, we see the following L^p-$L^{p'}$-estimate as a corollary.

Corollary A1. (L^p-$L^{p'}$-estimate) *Let k and l be nonnegative integeres, and let v belong to $C_o^\infty(R^N)$. Then, it holds that for $1\le p\le 2$ and $\gamma\in R$,*

$$\begin{aligned} &\|D^k_x D^l_t \mathcal{F}^{-1}\{\phi_i(\xi,t)\widehat{v}(\xi)\}\|_{H^\gamma_{p'}} \\ &\le C\,\Big\{(1+t)^{-(\frac{k}{2}+l+\frac{N}{2}(\frac{1}{p}-\frac{1}{2}))} + |t|^{-(N-1)(\frac{1}{p}-\frac{1}{2})}e^{-\nu t}\Big\}\|v\|_p \end{aligned} \tag{2.16}$$

with some $0 < \nu < 1/2$, *provided that* $k + l + \gamma + (N+1)(1/p - 1/2) \leq m + i - 1$, $i = 1, 2$.

Applying Theorem A1 to the equation (2.1) or using similar argument as in the proof of Theorem A1, we can obtain the desired L^q-estimates (with some $2 \leq q \leq \infty$) of the solution $u(t)$ and its derivatives $D_x^k D_t^l u(t)$ of the equation (2.1).

THEOREM A2. *Let* $(u_0, u_1) \in H^{k+l} \cap L^r \times H^{[k+l-1]^+} \cap L^r$, k *and* l *being nonnegative integeres,* $1 \leq r \leq 2$, *and let* f *be an appropriate function such that the right hand side of* (2.17c) *is bounded. Suppose that* $u(t)$ *is a solution of the equation* (2.1). *Then, we see that*

(2.17a)

$$\begin{aligned}&\|D_x^k D_t^l u(t)\| \\ &\leq \|\mathcal{F}^{-1}\{(\sqrt{-1}\,\xi)^k (\frac{d}{dt})^l \widehat{u_L}(\xi, t)\}\| + \|\mathcal{F}^{-1}\{(\sqrt{-1}\,\xi)^k (\frac{d}{dt})^l \widehat{I_f}(\xi, t)\}\| \\ &\equiv K_1(t) + K_2(t),\end{aligned}$$

and the terms $K_1(t)$ *and* $K_2(t)$ *have the following estimates :*

(i) *The term* $K_1(t)$ *has that*

$$\begin{aligned} K_1(t) &\leq C\,(1+t)^{-(\frac{k}{2}+l+\frac{N}{2}(\frac{1}{r}-\frac{1}{2}))}\big(\|u_0\|_r + \|u_1\|_r\big) \\ &\quad + C\,e^{-\nu t}\big(\|u_0\|_{H^{k+l}} + \|u_1\|_{H^{[k+l-1]^+}}\big) \end{aligned} \tag{2.17b}$$

with some $0 < \nu < 1/2$.

(ii) *The term* $K_2(t)$ *has that for* $1 \leq p \leq 2$, $0 \leq i \leq k$, and $m \geq [k + l - 1]^+$ being a integer,

$$K_2(t) \leq C \sum_{j=1}^{l-1} \|D_x^k D_t^{l-j-1} f(t)\|_{H^s}, \quad s = \begin{cases} j-1 & \textit{if } \ j \textit{ is odd}, \\ j-2 & \textit{if } \ j \textit{ is even}, \end{cases}$$

(2.17c)

$$\begin{aligned} &+ C \int_0^t (1+t-s)^{-(\frac{k-i}{2}+l+\frac{N}{2}(\frac{1}{p}-\frac{1}{2}))} \|D_x^i f(s)\|_p ds \\ &+ C \int_0^t e^{-\nu(t-s)} \|D_x^m f(s)\|\, ds \\ &\Big(\equiv K_2^{(1)}(t) + K_2^{(2)}(t) + K_2^{(3)}(t) \Big) \end{aligned}$$

with some $0<\nu<1/2$*, where we exclude the term* $K_2^{(1)}(t)$ *from* (2.17c) *if* $l \le 1$.

REMARK 2.1. When $k=l=0$, we have already known the same results as above from [10]. We can give the proof of this theorem easier than the proof of the following one for L^q-estimate. So, we omit the proof of this theorem (but we give the proof of following theorem).

THEOREM A3. *Let* $(u_0,u_1)\in H^{k+l}\cap L^r\times H^{[k+l-1]^+}\cap L^r$, k *and* l *being nonnegative interes,* $1\le r\le 2$*, and let* f *be an appropriate function such that the right hand side of* (2.18d) *is bounded. Suppose that* $u(t)$ *is a solution of the equation* (2.1)*. Then, we see that for* $2\le q\le\infty$,

(2.18a)

$$\begin{aligned}
&\|D_x^k D_t^l u(t)\|_q \\
&\le \|\mathcal{F}^{-1}\{(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\widehat{u_L}(\xi,t)\}\|_q+\|\mathcal{F}^{-1}\{(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\widehat{I_f}(\xi,t)\}\|_q \\
&\equiv L_1(t)+L_2(t)\,,
\end{aligned}$$

and the terms $L_1(t)$ *and* $L_2(t)$ *have the following estimates :*

(i) *The term* $L_1(t)$ *has that*

$$\begin{aligned}
L_1(t) &\le C\,(1+t)^{-(\frac{k}{2}+l+\frac{N}{2}(\frac{1}{r}-\frac{1}{q}))}\big(\|u_0\|_r+\|u_1\|_r\big) \\
&\quad + C\,e^{-\nu t}\big(\|u_0\|_{H^{k+l}}+\|u_1\|_{H^{[k+l-1]^+}}\big)
\end{aligned} \tag{2.18b}$$

with some $0<\nu<1/2$*, provided that*

$$\frac{k+l}{N}\ge\frac{1}{2}-\frac{1}{q}\quad\Big(\frac{k+l}{N}>\frac{1}{2}\text{ if } q=\infty\Big)\,. \tag{2.18c}$$

(ii) *The term* $L_2(t)$ *has that for* $l\le 3$, $0\le i\le k$, $1\le p_1, p\le 2$*, and*

$m \geq 0$ being a integer.

$$
\begin{aligned}
L_2(t) \leq & \sum_{j=1}^{l-1} \|D_x^k D_t^{l-j-1} f(t)\|_q \\
& + C \int_0^t (1+t-s)^{-(\frac{k-i}{2}+l+\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q}))} \|D_x^i f(s)\|_{p_1} ds \\
& + C \int_0^t e^{-\nu(t-s)} |t-s|^{-(N-1)(\frac{1}{p}-\frac{1}{2})} \|D_x^m f(s)\|_p ds \\
\Big(\quad & \equiv L_2^{(1)}(t) + L_2^{(2)}(t) + L_2^{(3)}(t) \quad \Big)
\end{aligned}
\tag{2.18d}
$$

with some $0 < \nu < 1/2$, provided that there exists $\gamma > 0$ such that

$$
\left\{
\begin{aligned}
& 1 - \frac{1}{p} - \frac{\gamma}{N} \leq \frac{1}{q} \leq 1 - \frac{1}{p} \quad \Big(1 - \frac{1}{p} - \frac{\gamma}{N} < 0 \text{ if } q = \infty\Big), \\
& k + l + \gamma + (N+1)\Big(\frac{1}{p} - \frac{1}{2}\Big) \leq m + 1,
\end{aligned}
\right.
\tag{2.18e}
$$

where we exclude the term $L_2^{(1)}(t)$ from (2.18d) if $l \leq 1$.

To proof of these theorem, we use the following : From (2.4), we see that

$$
\Big(\frac{d}{dt}\Big)^l \phi_2(\xi, t)\Big|_{t=0} = \frac{1}{2^l} \sum_{j=0}^{l} \binom{l}{j} (-1)^j \{\lambda^{l-j-1} + (-\lambda)^{l-j-1}\}. \tag{2.19a}
$$

In particular, we see that for $l = 1, 2, 3$,

$$
\phi_2(\xi, 0) = 0, \quad \Big(\frac{d}{dt}\Big)^1 \phi_2(\xi, t)\Big|_{t=0} = 0, \quad \Big(\frac{d}{dt}\Big)^2 \phi_2(\xi, t)\Big|_{t=0} = -1, \tag{2.19b}
$$

respectively. Furthermore, we see from (2.3c) and (2.19) that

$$
\begin{aligned}
(\frac{d}{dt})^1 \widehat{I}_f(\xi,t) &= \int_0^t (\frac{d}{dt})^1 \phi_2(\xi,t-s)\widehat{f}(\xi,s)ds\,,\\
(\frac{d}{dt})^2 \widehat{I}_f(\xi,t) &= \widehat{f}(\xi,t) + \int_0^t (\frac{d}{dt})^2 \phi_2(\xi,t-s)\widehat{f}(\xi,s)ds\,,
\end{aligned}
\tag{2.20}
$$

$$
(\frac{d}{dt})^3 \widehat{I}_f(\xi,t) = (\frac{d}{dt})^1 \widehat{f}(\xi,t) - \widehat{f}(\xi,t) + \int_0^t (\frac{d}{dt})^3 \phi_2(\xi,t-s)\widehat{f}(\xi,s)ds\,,
$$

and for $l \geq 4$

$$
\begin{aligned}
(\frac{d}{dt})^l \widehat{I}_f(\xi,t) &= (\frac{d}{dt})^{l-2} \widehat{f}(\xi,t) - (\frac{d}{dt})^{l-3} \widehat{f}(\xi,t)\\
&\quad + \sum_{j=3}^{l-1} (\frac{d}{dt})^j \phi_2(\xi,t)\Big|_{t=0} (\frac{d}{dt})^{l-j-1} \widehat{f}(\xi,t)\\
&\quad + \int_0^t (\frac{d}{dt})^l \phi_2(\xi,t-s)\widehat{f}(\xi,s)ds\,.
\end{aligned}
$$

Proof of Theorem A3. Now, we assume $2 \leq q < \infty$. (The case $q = \infty$ is treated quite similarly by a trivial modification.) First, we shall give the proof of (i) in Theorem A3. Recall that

$$
\widehat{u_L}(\xi,t) = \frac{1}{2}\big(\phi_1(\xi,t) + \phi_2(\xi,t)\big)\widehat{u_0}(\xi) + \phi_2(\xi,t)\widehat{u_1}(\xi)\,.
$$

From the Sobolev embedding theorem

$$
H^{k+l} \subset L^q \quad \text{if} \quad \frac{k+l}{2} \geq \frac{1}{2} - \frac{1}{q}\,, \tag{2.21}
$$

we have that

(2.22)

$$
\begin{aligned}
L_1(t) \le \sum_{j=1}^{3} \|\mathcal{F}^{-1}\Big\{\chi_j(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l \\
\Big(\frac{1}{2}\big(\phi_1(\xi,t)+\phi_2(\xi,t)\big)\widehat{u_0}(\xi)+\phi_2(\xi,t)\widehat{u_1}(\xi)\big)\Big)\Big\}\|_q \\
\le \|\mathcal{F}^{-1}\Big\{\chi_1(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l \\
\Big(\frac{1}{2}\big(\phi_1(\xi,t)+\phi_2(\xi,t)\big)\widehat{u_0}(\xi)+\phi_2(\xi,t)\widehat{u_1}(\xi)\big)\Big)\Big\}\|_q \\
+C\sum_{j=2}^{3}\Big\{\|\mathcal{F}^{-1}\{\chi_j(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l\phi_1(\xi,t)\widehat{u_0}(\xi)/2\}\|_{H^{k+l}} \\
+\|\mathcal{F}^{-1}\{\chi_3(\xi)(\sqrt{-1}\,\xi)^k\big(\frac{d}{dt}\big)^l \\
\phi_2(\xi,t)\big(\widehat{u_0}(\xi)/2+\widehat{u_1}(\xi)\big)\}\|_{H^{[k+l-1]^+}}\Big\} \\
\le C\,(1+t)^{-(\frac{k}{2}+l+\frac{N}{2}(\frac{1}{r}-\frac{1}{q}))}\big(\|u_0\|_r+\|u_1\|_r\big) \\
+C\,e^{-\nu t}\big(\|u_0\|+\|u_1\|\big)+C\,e^{-\nu t}\big(\|u_0\|_{H^{k+l}}+\|u_1\|_{H^{[k+l-1]^+}}\big)\,,
\end{aligned}
$$

using Theorem A1, which give the desired estimate (2.18b-c).

Next, we shall give the proof of (ii) in Theorem A3 the case when $l=3$ only. (The case $l\le 2$ can be proved in the same way.) From the Gagliardo-Nirenberg inequality

$$
(2.23)\qquad H^{\gamma}_{p'}\subset L^q \quad\text{if}\quad 1-\frac{1}{p}-\frac{\gamma}{N}\le\frac{1}{q}\le 1-\frac{1}{p},\ \gamma>0\,,
$$

and (2.20), we obtain that

$$
\begin{aligned}
L_2(t) &\equiv \|\mathcal{F}^{-1}\{(\sqrt{-1}\xi)^k(\frac{d}{dt})^l \widehat{I}_f(\xi,t)\}\|_q \\
&\le \|D_x^k D_t^{l-2} f(t)\|_q + \|D_x^k D_t^{l-3} f(t)\|_q \\
&\quad + C\int_0^t \Big\{ \|\mathcal{F}^{-1}\{\chi_1(\xi)(\sqrt{-1}\xi)^k(\frac{d}{dt})^l \phi_2(\xi,t-s)\widehat{f}(\xi,s)\}\|_q \\
&\qquad + \|\mathcal{F}^{-1}\{\chi_2(\xi)(\sqrt{-1}\xi)^k(\frac{d}{dt})^l \phi_2(\xi,t-s)\widehat{f}(\xi,s)\}\|_{H^{\gamma_2}_{p_2'}} \\
&\qquad + \|\mathcal{F}^{-1}\{\chi_3(\xi)(\sqrt{-1}\xi)^k(\frac{d}{dt})^l \phi_2(\xi,t-s)\widehat{f}(\xi,s)\}\|_{H^{\gamma}_{p'}} \Big\} ds
\end{aligned}
$$

(2.24)

$$
\begin{aligned}
&\le \sum_{j=1}^{l-1} \|D_x^k D_t^{l-j-1} f(t)\|_q \\
&\quad + C\int_0^t \Big\{ (1+t-s)^{-(\frac{k-i}{2}+l+\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q}))} \|D_x^i f(s)\|_{p_1} \\
&\qquad + e^{-\nu(t-s)}\|D_x^{i_2} f(s)\|_{p_2} + e^{-\nu(t-s)}|t-s|^{-(N-1)(\frac{1}{p}-\frac{1}{q})} \\
&\qquad\qquad \|D_x^m f(s)\|_p \Big\} ds
\end{aligned}
$$

for any $0 \le i \le k$, $i_2 \ge 0$, using Theorem A1, where m is a nonnegative integer, p_1 is any number with $1 \le p_1 \le 2$, (γ_2, p_2) should satisfy the condition (2.23), and (γ, p) should satisfy the condition (2.23) and (2.8b). When $1 \le p \le 2$ and $2 \le q < \infty$, there always exists γ satisfying (2.23). Hence, we can take $p_2 = p_1$ $(1 \le p_1 \le 2)$ in (2.24). Furthermore, we take $i_2 = i$ $(0 \le i \le k)$. Thus, from these, we get the desired estimate (2.18d-e). The proof of Theorem A3 is now complete. (We can prove Theorem A2 easier than Theorem A3.) □

B. Semilinear Dissipative Wave Equation

3. Hypotheses and Main Results

In this section and the following sections, we consider the decay property of the solution to the Cauchy problem for the semilinear wave equation with a dissipative term :

$$(3.1)\qquad \begin{cases} u_{tt} - \Delta u + u_t + f(u) = 0 & \text{in} \quad R^N \times [0,\infty), \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x). \end{cases}$$

Now, we state our hypotheses on the nonlinear term $f(u)$ in the following.

HYP. 0. *$f(u)$ is a continuous function on R and satisfies the conditions*

$$(3.2a)\qquad f(u)u \geq kF(u) \geq 0, \quad F(u) \equiv 2\int_0^u f(\eta)d\eta,$$

for some $k > 0$ and

$$(3.2b)\qquad |f(u)| \leq k_0|u|^{\alpha+1}$$

for some $k_0 > 0$ and $\alpha > 0$.

HYP. m. $(m = 1,2,\cdots)$ *$f(u)$ belongs to $C^m(R)$ and satisfies the condition*

$$(3.3.m)\qquad |f^{(m)}(u)| \leq k_m|u|^{[\alpha+1-m]^+}$$

for some $k_m > 0$, $m = 1,2,\cdots$, and $\alpha > 0$.

We shall pick up freely appropriate set of hypotheses on the nonlinear term $f(u)$ from HYP. m, $m = 0,1,2,\cdots$.

Using some results which have given by [10] (see, e.g., Proposition 4.1 in the following sections), we shall give our main results in this paper by the following two theorems.

First theorem is the following :

THEOREM B1. *Let $1 \leq N \leq 3$ and $(u_0, u_1) \in H^1 \cap L^r \times L^2 \cap L^r$, $1 \leq r \leq 2$, and let* HYP. 0 *be satisfied with α such that*

$$(3.4a)\qquad 4/N \leq \alpha \leq 2/(N-2) \quad (4/N \leq \alpha < \infty \text{ if } N = 1,2).$$

Then, the solution $u(t) \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ of the equation (3.1) satisfies that for $0 \leq k, l, k+l \leq 1$,

$$(3.4b)\qquad \|D_x^k D_t^l u(t)\| \leq C_{[1]}(1+t)^{-\omega_{k,l}-\eta},$$

where $\omega_{k,l}$ and η are nonnegative numbers given by (0.4), (0.5), *respectively, and $C_{[1]}$ is a certain positive constant given by* (0.6) (*see Notation in* §0).

REMARK 3.1. For $N \geq 1$, $\alpha > 0$, and $(u_0, u_1) \in H^1 \times L^2$, weak solutions $u(t) \in L^\infty([0,\infty); H^1 \cap L^{\alpha+2}) \cap W^{1,\infty}([0,\infty); L^2)$ of the equation (3.1), which be established the existence by Strauss [22], have the boundedness and decay property that

$$\|u(t)\| \leq C_{[1]} \tag{3.5a}$$

and

(3.5b)

$$E_1(t) \equiv \|D_x u(t)\|^2 + \|D_t u(t)\|^2 + \int_{R^N} F(u(t))\, dx \leq C_{[1]}(1+t)^{-1/2},$$

where $C_{[1]}$ is a certain positive constant depending on $\|u_0\|_{H^1} + \|u_0\|_{\alpha+1} + \|u_1\|$. This fact have already been given in [10] and the reference.

Second theorem is the following :

THEOREM B2. *Let $1 \leq N \leq 6$ and $(u_0, u_1) \in H^{m+1} \cap L^r \times H^m \cap L^r$, $m = 1, 2$, $1 \leq r \leq 2$, and let* HYP. 0 $\sim$ HYP. m *be satisfied with α such that*

$$4/N \leq \alpha < 4/[N-2]^+ . \tag{3.6a}$$

Then, the solution $u(t) \in \bigcap_{j=0}^{m+1} C^j([0,\infty); H^{m+1-j})$ of the equation (3.1) *satisfies that for $0 \leq k, l, k+l \leq m+1$,*

$$\|D_x^k D_t^l u(t)\| \leq C_{[m+1]}(1+t)^{-\theta_{k,l}} \tag{3.6b}$$

with

$$\theta_{k,l} = \begin{cases} \omega_{k,l} + \eta & \text{if } \ l \leq m\,, \\ \omega_{1,m} + \eta & \text{if } \ l = m+1\,. \end{cases} \tag{3.6c}$$

Moreover, it follows that for $2 \leq q \leq \infty$ ($2 \leq q < \infty$ if $N = 6$),

$$\|u(t)\|_q \leq C_{[3]}(1+t)^{-A_q} \tag{3.6d}$$

with

$$A_q = \frac{N}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \eta\,. \tag{3.6e}$$

REMARK 3.2. When $N = 7$, the result of Theorem B2 holds under a restricted condition $4/N \le \alpha \le 2N(N-1)/(N-2)(N-3)$ $(< 4/(N-2))$.

REMARK 3.3. Even in the case $0 < \alpha < 4/N$, we can derive some decay rates. Indeed, let $(u_0, u_1) \in H^1 \times L^2$, and let HYP. 0 $\sim$ HYP. m, $m = 1, 2$, be satisfied with α such that

$$\begin{cases} 0 < \alpha < 4/N & \text{if} \quad N \le 7\,, \\ 0 < \alpha \le 2N/(N-2)(N-3) & \text{if} \quad N \ge 8\,. \end{cases} \tag{3.7a}$$

Then, the decay estimate (3.6b) holds true with $\theta_{k,l}$, $0 \le k, l, k+l \le m+1$, replaced by

$$\widetilde{\theta}_{k,l} = \begin{cases} \omega_{1,0} + [k+l-1]^+\omega + l\widetilde{\omega} & \text{if} \quad l \le m\,, \\ \omega_{1,0} + (l-1)\omega + (l-1)\widetilde{\omega} & \text{if} \quad l = m+1\,, \end{cases} \tag{3.7b}$$

where ω is given by
(3.7c)

$$\omega = \begin{cases} \alpha/8 & \text{if} \quad N = 1\,, \\ \alpha/4 - \varepsilon & \text{if} \quad N = 2\,, \\ \alpha/(4-(N-2)\alpha) & \text{if} \quad N \ge 3 \text{ and } \alpha \le 2/(N-2)\,, \\ \alpha/(4-(N-2)\alpha) - \varepsilon & \text{if} \quad N \ge 3 \text{ and } \alpha > 2/(N-2) \end{cases}$$

for any $0 < \varepsilon \ll 1$, and $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = \begin{cases} [N\alpha - 2]^+/4 & \text{if} \quad \alpha < 2/(N-2)\,, \\ \omega & \text{if} \quad \alpha \ge 2/(N-2)\,. \end{cases} \tag{3.7d}$$

We note that $0 \le \widetilde{\omega} \le \omega < \omega_{1,0}$ and $\eta = 0$ if $\alpha < 4/N$.

REMARK 3.4. When $1 \le N \le 5$ and $(u_0, u_1) \in H^{m+1} \cap L^r \times H^m \cap L^r$, $m \ge 3$, $1 \le r \le 2$, we can derive some decay estimates of the solution and its derivatives of the equation (3.1) under the assumption which $f(u)$ is an m-times continuously differentiable function and satisfies HYP. 0 $\sim$ HYP. 2 with α such that

$$\left\{ \begin{array}{lll} 4/N \le \alpha < \infty & \text{if} & N = 1\,, \\ 0 < \alpha < 4/[N-2]^+ & \text{if} & N \ge 2\,. \end{array} \right. \tag{3.8a}$$

Indeed, we can see that the solution $u(t) \in \bigcap_{j=0}^{m+1} C^j([0,\infty); H^{m+1-j})$ of the equation (3.1) satisfies that for $0 \le k, l, k+l \le m+1$,

$$\|D_x^k D_t^l u(t)\| \le C_{[m+1]}(1+t)^{-\theta_{k,l}} \tag{3.8b}$$

with

$$\theta_{k,l} = \left\{ \begin{array}{lll} \omega_{k,l} + \eta & \text{if} & N = 1,2 \text{ and } l \le 2\,, \\ & \text{or} & N \ge 3 \text{ and } l \le 3\,, \\ \omega_{k+l-2,2} + \eta & \text{if} & N = 1,2 \text{ and } l \ge 3\,, \\ \omega_{k+l-3,3} + \eta & \text{if} & N \ge 3 \text{ and } l \ge 4\,. \end{array} \right. \tag{3.8c}$$

To derive these, we use the following fact that

$$|f^{(i)}(u(t))| \le C_{[3]}(\|u(t)\|_\infty)|u(t)|^{3-i}\,, \quad i = 0,1,2,3\,. \tag{3.8d}$$

(Cf. Consider the Taylor expansion of $f(u)$ at $u = 0$, noting $\|u(t)\|_\infty \le C_{[3]}$. See [10].)

4. Proof of Theorem B1

In this section we shall prove Theorem B1. To this we use the following results which have been given in [10]. (We omit here the proof.)

PROPOSITION 4.1. *Under the assumption of Theorem B1, the solution $u(t)$ of the equation* (3.1) *satisfies that*

$$\|u(t)\| \le C_{[1]}(1+t)^{-\eta} \quad \text{and} \quad \|D_x u(t)\| + \|D_t u(t)\| \le C_{[1]}(1+t)^{-1/2-\eta}\,. \tag{4.1}$$

his proposition implies Theorem B1 with $(k,l)=(0,0),(1,0)$. For case $(k,l)=(0,1)$ in Theorem B1, we need improve the decay nate of $\|D_t u(t)\|$. Now, Applying Theorem A2 with $k=0, l=$ $= r, m=0$, we have that

$$u(t)\| \le C_{[1]}(1+t)^{-\omega_{0,1}-\eta_r^*} + C\int_0^t (1+t-s)^{-\omega_{0,1}-\eta_r^*}\|f(u(s))\|_r ds$$

)

$$+C\int_0^t e^{-\nu(t-s)}\|f(u(s))\|ds\,.$$

:, we see from Proposition 4.1 that for $4/N \le \alpha \le 2/(N-2)$ $V \le \alpha < \infty$ if $N=1,2$),

$$(u(t))\|_r \le C\|u(t)\|_{r(\alpha+1)}^{\alpha+1} \le C\|u(t)\|^{(\alpha+1)(1-\xi_1)}\|D_x u(t)\|^{(\alpha+1)\xi_1}$$

a) $$\le C_{[1]}(1+t)^{-\sigma_1}\,,$$

re $\xi_1 = N(1/2 - 1/r(\alpha+1))$ and

$$\sigma_1 = (\alpha+1)(1-\xi_1)\eta + (\alpha+1)\xi_1(\omega_{1,0}+\eta)$$

b) $$= N\alpha/4 + \alpha\eta > \omega_{0,1}+\eta\,.$$

the other hand, we see

$$f(u(t))\| \le C\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \le C\|u(t)\|^{(\alpha+1)(1-\xi_2)}\|D_x u(t)\|^{(\alpha+1)\xi_2}$$

a) $$\le C_{[1]}(1+t)^{-\sigma_2}\,,$$

re $\xi_2 = N(1/2 - 1/2(\alpha+1))$ and

b) $$\sigma_2 = N\alpha/4 + (\alpha+1)\eta > \omega_{0,1}+\eta\,.$$

s, applying Lemma 1.3 to the inequality (4.2), we obtain from (4.3) (4.4) that

) $$\|D_t u(t)\| \le C_{[1]}(1+t)^{-\omega_{0,1}-\eta}\,,$$

ch implies Theorem B1 with $(k,l)=(0.1)$. The proof of Theorem is now complete.

5. Proof of Theorem B2

At the first step, we shall prove Theorem B2 with $m = 1$. To this, we use the following results which have been given in [10]. (We omit the proof.)

PROPOSITION 5.1. *Under the assumption of Theorem B2 with $m = 1$, we have :*

(i). *The solution $u(t)$ of the equation* (3.1) *satisfies that*

(5.1)
$$\|u(t)\| \leq C_{[2]}(1+t)^{-\eta} \quad \text{and} \quad \|D_x u(t)\| + \|D_t u(t)\| \leq C_{[2]}(1+t)^{-1/2-\eta}.$$

(ii) *Moreover, it satisfies that for $3 \leq N \leq 6$ and $2 \leq q \leq q_*$,*

$$\|u(t)\|_q \leq C_{[2]}(1+t)^{-A_q}, \tag{5.2a}$$

where we set

$$q_* = \frac{2N(N-1)}{[(N-2)(N-3)]^+} + \varepsilon \tag{5.2b}$$

with $0 < \varepsilon \ll 1$, and

$$A_q = \frac{N}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \eta \tag{5.2c}$$

with η given by (0.5).

REMARK 5.1. The (ii) of Proposition 5.1 plays essential role in the previous paper [10] (and, of course, in this paper). For the proof of this proposition, we use Proposition 6.2 in Appendix and Theorem A3 with $k = l = 0$ and (3.5) in Remark 3.1.

Using Proposition 5.1, we see the following :

PROPOSITION 5.2. *Under the assumption of Theorem B2 with $m = 1$, the solution $u(t)$ of the equation* (3.1) *satisfies that for $(k, l) = (0,1), (2,0), (1,1), (0,2)$,*

$$\|D_x^k D_t^l u(t)\| \leq C_{[2]}(1+t)^{-\omega_{2,0}-\eta}. \tag{5.3}$$

oof of Proposition 5.2. First, applying Theorem A2 with $k =$ $0, i = 0, p = r, m = 1$, we have that

$$\iota(t)\| \leq C_{[2]}(1+t)^{-\omega_{2,0}-\eta_r^*} + C\int_0^t (1+t-s)^{-\omega_{2,0}-\eta_r^*}\|f(u(s))\|_r ds$$

$$+ C\int_0^t e^{-\nu(t-s)}\|D_x f(u(s))\|\, ds\,.$$

, using the results of (5.2) in Proposition 5.1, we see from HYP. 0 HYP. 1 that

$$\|f(u(t))\|_r \leq C\,\|u(t)\|_{r(\alpha+1)}^{\alpha+1} \leq C_{[2]}(1+t)^{-\sigma_1}\,, \tag*{.)}$$

e $\sigma_1 = (\alpha+1)A_{r(\alpha+1)} = N\alpha/4 + \alpha\eta > \omega_{2,0} + \eta$, and

$$\begin{aligned}\|D_x f(u(t))\| &\leq C\,\||u(t)|^\alpha D_x u(t)\| \leq C\,\|u(t)\|_{N\alpha}^\alpha \|D_x^2 u(t)\| \\ &\leq C_{[2]}(1+t)^{-\sigma_2}\|D_x^2 u(t)\|\,,\end{aligned} \tag*{)}$$

e $\sigma_2 = \alpha A_{N\alpha} = (N\alpha-2)/4 + \alpha\eta > 0$. Thus, applying Lemma 1.3 .4), we obtain from (5.4) and (5.5a-b) that

$$\|D_x^2 u(t)\| \leq C_{[2]}(1+t)^{-\omega_{2,0}-\eta}\,,$$

h gives the case $(k,l) = (2,0)$ in (5.4).
ext, we use that

$$\|D_x f(u(t))\| \leq C\,\|u(t)\|_{N\alpha}^\alpha \|D_x^2 u(t)\| \leq C_{[2]}(1+t)^{-\sigma_3}$$

$\sigma_3 = \alpha A_{N\alpha} + \omega_{2,0} + \eta > \omega_{2,0} + \eta$ to get the decay estimate for $\iota(t)\|, \|D_x D_t u(t)\|$, and $\|D_t^2 u(t)\|$. For these norm, we can get the ed estimates by a same way with (5.6) replaced by (5.7). □

loreover, using the Gagliardo-Nirenberg inequality, we see the folng (which improves the (ii) of Proposition 5.1).

OROLLARY 5.3. *Under the assumption of Theorem B2 with $m =$ he solution $u(t)$ of the equation* (3.1) *satisfies that for* $2 \leq q \leq$ $[N-4]^+$ $(2 \leq q < \infty$ *if* $N = 4)$,

$$\|u(t)\|_q \leq C_{[2]}(1+t)^{-A_q} \tag*{)}$$

with A_q given by (5.2c) *in Proposition 5.1.*

Next, by use Corollary 5.3, we shall improve the decay estimates for the case $(k,l)=(1,1),(0,2)$ in Proposition 5.2.

PROPOSITION 5.4. *Under the assumption of Theorem B2 with $m=1$, the solution $u(t)$ of the equation* (3.1) *satisfies that for $(k,l)=(1,1),(0,2)$,*

$$\|D_x^k D_t^l u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,1}-\eta}. \tag{5.9}$$

Proof of Proposition 5.4. Setting $v=Du$ ($D=D_x$ or D_t), $v(x,t)$ satisfies

$$v_{tt}-\Delta v+v_t=-Df(u). \tag{5.10a}$$

It is easy to see that Lemma 2.1 is applicable to this $v(x,t)$ with

$$E_2(t)\equiv \|D_x v(t)\|^2+\|D_t v(t)\|^2. \tag{5.10b}$$

In particular, setting $v=D_t u$, we have from HYP. 1 and Corollary 5.3 with $q=N\alpha$ that

(5.11)
$$\|D_t f(u(t))\|^2 \le C\,\|u(t)\|_{N\alpha}^{2\alpha}\|D_x D_t u(t)\|^2 \le C_{[2]}(1+t)^{-2b_1}E_2(t)$$

with $b_1=\sigma_2=(N\alpha-2)/4+\alpha\eta>1/2$ (by $\alpha>4/N$, cf. (5.5b)). While, we see from Proposition 5.2 with $(k,l)=(0,1)$ that

$$\|D_t u(t)\|^2 \le C_{[2]}(1+t)^{-2a} \tag{5.12}$$

with $a=\omega_{0,1}+\eta$ (we note that $\omega_{2,0}=\omega_{0,1}$). Thus, applying Lemma 2.1, we obtain from (5.11) and (5.12) that

$$E_{1+1}(t)\equiv \|D_x D_t u(t)\|^2+\|D_t^2 u(t)\|^2 \le C_{[2]}(1+t)^{-2\theta_{1,1}}, \tag{5.13a}$$

where

$$\theta_{1,1}=\min\Big\{\frac{1}{2}+a\,,\,a+d\Big\}=\frac{1}{2}+\omega_{0,1}+\eta=\omega_{1,1}+\eta, \tag{5.13b}$$

which is the desired estimates (5.9). □

Summing up the above Propositions 5.1, 5.2, and 5.3, we arrive at Theorem B2 with $m = 1$. The proof of Theorem B2 with $m = 1$ is now complete.

At the second step, we shall prove Theorem B2 with $m = 2$. First, we shall prove the following.

PROPOSITION 5.5. *Under the assumption of Theorem B2 with $m = 1$, the solution $u(t)$ of the equation (3.1) satisfies that for $k, l \geq 0$ such that $k + l = 3$,*

$$\|D_x^k D_t^l u(t)\| \leq C_{[3]}(1+t)^{-\theta_{k,l}} . \tag{5.14a}$$

with

$$\theta_{k,l} = \begin{cases} \omega_{k,l} + \eta & \text{if} \quad l \leq 1 , \\ \omega_{2,1} + \eta & \text{if} \quad l \geq 2 . \end{cases} \tag{5.14b}$$

Proof of Proposition 5.5. Setting $w = Dv$ and $v = Du$ ($D = D_x$ or D_t), $w(x,t)$ satisfies.

$$w_{tt} - \Delta w + w_t = -D^2 f(u) . \tag{5.15a}$$

Now, we note that $w = D^2 u$ can be regarded as $(N+1) \times (N+1)$ matrix valued function and it is clear that Lemma 2.1 is applicable to this $w(x,t)$ with

$$E_3(t) \equiv \|D_x w(t)\|^2 + \|D_t w(t)\|^2 . \tag{5.15b}$$

By HYP. 0 $\sim$ HYP. 2, we have

(5.16)

$$\|D^2 f(u(t))\|^2 \leq C\left\{\| \, |u(t)|^\alpha w(t)\|^2 + \| \, |u(t)|^{[\alpha-1]^+} v(t)^2 \|^2\right\} \equiv I_1(t)^2 + I_2(t)^2 .$$

Here, since the term $I_1(t)^2$ is treated by the same way as in (5.11), we see that

$$I_1(t)^2 \leq C_{[2]}(1+t)^{-2\delta_1} E_3(t) \tag{5.17}$$

with $b_1 = (N\alpha - 2)/4 + \alpha\eta > 1/2$. Next, we shall estimate the term $I_2(t)^2$. When $N = 4, 5, 6$, we have

(5.18a)
$$I_2(t)^2 \le C\,\|u(t)\|_{2N/(N-2)}^{[\alpha-1]^+}\|v(t)\|_p^4$$
$$\le C\,\|D_x u(t)\|^{[\alpha-1]^+}\|D_x v(t)\|^{4(1-\theta)}\|D_x^2 v(t)\|^{4\theta}$$

with $2/p = 1/2 - (N-2)[\alpha-1]^+/2N$ and $\theta = N(1/p - (N-2)/2N)$. (When $N \le 3$, trivial modifications are needed. We can use $\|u(t)\|_\infty \le C_{[2]} < \infty$. Cf. (5.8) in Corollary 5.3.)

First, in particular, we set $w = D_x v = D_x^2 u$ in (5.15). Then, we see from (5.15) and (5.18a) that

$$I_2(t)^2 \le C\,\|D_x u(t)\|^{[\alpha-1]^+}\|D_x^2 u(t)\|^{4(1-\theta)}E_3(t)^{2\theta}$$
(5.18b)
$$\le C_{[2]}(1+t)^{-2b_2}E_3(t)^{\mu}\,,$$

using Propositions 5.1 and 5.2, where

(5.18c)
$$b_2 = \frac{1}{2}\left\{[\alpha-1]^+(\omega_{1,0}+\eta) + 4(1-\theta)(\omega_{2,0}+\eta)\right\}$$

and

(5.18d)
$$\mu = 2\theta = \frac{N-4}{2} + \frac{N-2}{4N}[\alpha-1]^+ \quad (0 \le \mu \le 1)\,,$$

and we note that (if $a = \omega_{2,0} + \eta$)

(5.18e)
$$\frac{b_2}{[1-\mu]^+} \ge \frac{a+b_2}{2-\mu} \ge \omega_{3,0} + \eta\,.$$

Thus, we obtain from (5.16), (5.17), and (5.18) that

(5.19)
$$\|D_x^2 f(u(t))\|^2 \le C_{[2]}(1+t)^{-2b_1}E_3(t) + C_{[2]}(1+t)^{-2b_2}E_3(t)^{\mu}\,.$$

While, we see from Proposition 5.2 with $(k,l) = (2,0)$ that

(5.20)
$$\|w(t)\|^2 = \|D_x^2 u(t)\|^2 \le C_{[2]}(1+t)^{-2a_1}$$

with $a_1 = \omega_{2,0} + \eta$. Thus, applying Lemma 2.1 to (5.15), we obtain form (5.19) and (5.20) that

$$E_{3+0}(t) \equiv \|D_x^3 u(t)\|^2 + \|D_x^2 D_t u(t)\|^2 \leq C_{[3]}(1+t)^{-2\theta_{3,0}} \tag{5.21}$$

with $\theta_{3,0} = 1/2 + a_1 = \omega_{3,0} + \eta$.

Next, in particular, we set $w = D_x D_t u$ in (5.15). Then, we see from (1.18a) that

$$I_2(t)^2 \leq C_{[3]}(1+t)^{-2b_3} \tag{5.22}$$

with $b_3 \geq \omega_{1,2} + \eta$ ($> \omega_{2,1} + \eta$), using Propositions 5.1 and 5.2 and (5.21). Thus, we obtain from (5.16), (5.17), and (5.22) that

$$\|D_x D_t f(u(t))\|^2 \leq C_{[2]}(1+t)^{-2b_1} E_3(t) + C_{[3]}(1+t)^{-2b_3}\,. \tag{5.23}$$

While, we see from Proposition 5.4 with $(k, l) = (1, 1)$ that

$$\|w(t)\|^2 \equiv \|D_x D_t u(t)\|^2 \leq C_{[2]}(1+t)^{-2a_2} \tag{5.24}$$

with $a_2 = \omega_{1,1} + \eta$. Thus, applying Lemma 2.1 to (5.15), we obtain from (5.23) and (5.24) that

$$E_{2+1}(t) \equiv \|D_x^2 D_t u(t)\|^2 + \|D_x D_t^2 u(t)\|^2 \leq C_{[3]}(1+t)^{-2\theta_{2,1}} \tag{5.25}$$

with $\theta_{2,1} = 1/2 + a_2 = \omega_{2,1} + \eta$.

Finally, we set $w = D_t^2 u(t)$ in (5.15). Then, we can get the following estimate by the same way as the above

$$E_{1+2}(t) \equiv \|D_x D_t^2 u(t)\|^2 + \|D_t^3 u(t)\|^2 \leq C_{[3]}(1+t)^{-2\theta_{1,2}} \tag{5.26}$$

with $\theta_{1,2} = \omega_{2,1} + \eta$.

The above escalated energy estimates (5.21), (5.25), and (5.26) give the desired estimates (5.14). $\square$

From Proposition 5.5, we see immediately the following (which improves the (ii) of Proposition 5.1 and Corollary 5.3).

COROLLARY 5.6. *Under the assumption of Theorem B2 with $m = 2$, the solution $u(t)$ of the equation (3.1) satisfies that for $2 \le q \le 2N/[N-6]^+$ ($2 \le q < \infty$ if $N = 6$),*

$$\|u(t)\|_q \le C_{[3]}(1+t)^{-A_q} \tag{5.27}$$

with A_q given by (5.2c) in Proposition 5.1.

Finally of this section, by use Corollary 5.6, we shall improve the decay estimates for the case $(k,l) = (0,2), (1,2), (0,3)$ in Propositions 5.4 and 5.5.

PROPOSITION 5.7. *Under the assumption of Theorem B2 with $m = 2$, the solution $u(t)$ of the equation (3.1) satisfies that for $(k,l) = (0,2), (1,2), (0,3)$,*

$$\|D_x^k D_t^l u(t)\| \le C_{[3]}(1+t)^{-\theta_{k,l}} . \tag{5.28a}$$

with

$$\theta_{k,l} = \begin{cases} \omega_{k,l} + \eta & \text{if} \quad l \le 2 , \\ \omega_{1,2} + \eta & \text{if} \quad l = 3 . \end{cases} \tag{5.28b}$$

Proof of Proposition 5.7. By use the equation (5.10a) with $v = D_t u$, we see from (5.11), (5.13), and (5.14) that

$$\begin{aligned} \|D_t^2 u(t)\| &\le \|D_x^2 D_t u(t)\| + \|D_t^3 u(t)\| + \|D_t f(u(t))\| \\ &\le C_{[3]}(1+t)^{-a_3} , \quad a_3 = \omega_{0,2} + \eta , \end{aligned} \tag{5.29}$$

noting that $\omega_{2,1} = \omega_{0,2}$, which gives the case $(k,l) = (0,2)$ in Proposition 5.7. Next, to improve estimates of the case $(k,l) = (1,2), (0,3)$ in Proposition 5.5, we set $w = D_t^2 u$ in (5.15) and we shall estimate $\|D_t^2 f(u(t))\|$. By the same way as in (5.17) and (5.22), we see that

$$\|D_t^2 f(u(t))\|^2 \le C_{[2]}(1+t)^{-2b_1} E_3(t) + C_{[3]}(1+t)^{-2b_4} , \tag{5.30}$$

where $b_1 > 1/2$ and $b_4 = b_3 \ge \omega_{1,2} + \eta$. Thus, applying Lemma 2.1 to (5.15), we obtain from (5.29) and (5.30) that

$$E_{1+2}(t) \equiv \|D_x D_t^2 u(t)\|^2 + \|D_t^3 u(t)\|^2 \le C_{[3]}(1+t)^{-2\theta_{1,2}}$$

with $\theta_{1,2} = \omega_{1,2} + \eta$, which gives the desired estimates (5.28). □

Summing up the above Propositions 5.5 and 5.7, we arrive at Theorem B2 with $m = 2$. The proof of Theorem B2 is now finished.

6. Appendix

In this section, by way of precaution we shall give a sketch of the proof of Remark 3.3 with $m = 1$. To this we use the following two results given in [10].

The first one is the following :

PROPOSITION 6.1. *Let* $(u_0, u_1) \in H^2 \times H^1$, *and let* HYP.0 ~ HYP.1 *be satisfied with* α *such that*

$$\begin{cases} 0 < \alpha < 4/N & \text{if} \quad N \le 4, \\ 0 < \alpha \le 2/(N-2) & \text{if} \quad N \ge 5. \end{cases} \tag{6.1a}$$

Then, the solution $u(t)$ *of the equation* (3.1) *satisfies that for* $k, l \ge 0$ *such that* $k + l = 2$,

$$\|D_x^k D_t^l u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,0}-\omega}, \tag{6.1b}$$

where ω *is given by* (3.7c).

The proof of Proposition 6.1 is easy. (We omit here the proof. See [10].)

The second one is following :

PROPOSITION 6.2. *Let* $N \ge 3$ *and* $(u_0, u_1) \in H^2 \times H^1$, *and let* HYP.0 ~ HYP.1 *be satisfied with* α *such that*

$$\begin{cases} 2/(N-1) \le \alpha < 4/(N-2) & \text{if} \quad N \le 6, \\ 2/(N-1) \le \alpha \le 2N/(N-2)(N-3) & \text{if} \quad N \ge 7. \end{cases} \tag{6.2a}$$

Then, the solution $u(t)$ *of the equation* (3.1) *is uniformly bounded in* $L^{q_*}(R^N)$, i.e.,

$$\|u(t)\|_{q_*} \le C_{[2]} < \infty \tag{6.2b}$$

where q_* *is given by*

$$q_* = \frac{2N(N-1)}{[(N-2)(N-3)]^+} + \varepsilon \tag{6.2c}$$

with $0 < \varepsilon \ll 1$.

This result is one of the most important fact through this paper (and the previous paper [10]). So, we sketch the proof of Proposition 6.2.

Proof of Proposition 6.2. We utilize Theorem A3 with $k = l = 0$, $q = q_*$, $m = 1$, $p = p_* \equiv 2(N-1)/(N+1) + \varepsilon, 0 < \varepsilon \ll 1$, and HYP.0 $\sim$ HYP.1 to get

(6.3)
$$\begin{aligned}
&\|u(t)\|_{q_*}\\
&\le C_{[2]}(1+t)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q_*})} + C\int_0^t (1+t-s)^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q_*})}\|u(s)\|_{p_1(\alpha+1)}^{\alpha+1}ds\\
&\quad + C\int_0^t (t-s)^{-(N-1)(\frac{1}{p_*}-\frac{1}{2})}e^{-\nu(t-s)}\||u(s)|^\alpha D_x u(s)\|_{p_*}ds\,.
\end{aligned}$$

Here, we set

(6.4a)
$$p_1 = \begin{cases} 1 & \text{if} \quad \alpha > 1\,, \\ 2/(\alpha+1) & \text{if} \quad \alpha \le 1\,. \end{cases}$$

Then, we see

(6.4b)
$$\frac{N}{2}(\frac{1}{p_1} - \frac{1}{q_*}) > 1 \quad \text{and} \quad 2 \le p_1(\alpha+1) \le 2N/(N-2)\,.$$

Thus, we have from Theorem B1 and Remark 3.1 that

(6.5)
$$\|u(t)\|_{p_1(\alpha+1)} \le C\,\|u(t)\|_{H^1} \le C_{[1]} < \infty\,.$$

Also, if $2/(N-1) \le \alpha < 2N/(N-1)(N-2)$, we have

(6.7a)
$$\||u(t)|^\alpha D_x u(t)\|_{p_*} \le C\,\|u(t)\|_{H^1}^{1+2\beta_2}\|u(t)\|_{q_*}^{\mu_2} \le C_{[1]}\|u(t)\|_{q_*}^{\mu_2}\,,$$

where

(6.7b)
$$2\beta_2 = (\frac{1}{p_*} - \frac{1}{2} - \frac{\alpha}{q_*})(\frac{N-2}{2N} - \frac{1}{q_*})^{-1}$$

and

$$\mu_2 = \alpha - 2\beta_2 \quad (< 1 \text{ if } \alpha < 4/(N-2))\,, \tag{6.7c}$$

which follows from the Gagliardo-Nirenberg inequality. Thus, we obtain from (6.3), (6.5), (6.6), and (6.7) that

$$\begin{aligned}\|u(t)\|_{q_*} \le C_{[2]}(1+t)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q_*})} + C_{[1]}\int_0^t (1+t-s)^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q_*})}ds \\ + C_{[1]}\int_0^t (t-s)^{-(N-1)(\frac{1}{p_*}-\frac{1}{2})}e^{-\nu(t-s)}\big(1+\|u(s)\|_{q_*}^{\mu_2}\big)\,ds\end{aligned} \tag{6.8}$$

with $0 < \nu < 1/2$ and $0 \le \mu_2 < 1$, which implies (6.3) by Lemma 1.3. □

Noting Proposition 6.1, it is sufficiently that we give some decay estimate of $\|D_x^k D_t^l u(t)\|$, $k+l=2$, when $N \ge 5$ and $2/(N-2) < \alpha < 4/N$.

PROPOSITION 6.3. *Let $N \ge 5$ and $(u_0, u_1) \in H^2 \times H^1$, and let* HYP. 0 ~ HYP. 1 *be satisfied with α such that*

$$\begin{cases} 2/(N-2) \le \alpha < 4/N & \text{if } \ N \le 7\,, \\ 2/(N-2) \le \alpha \le 2N/(N-2)(N-3) & \text{if } \ N \ge 8\,. \end{cases} \tag{6.9a}$$

Then, the solution $u(t)$ of the equation (3.1) satisfies that for $k, l \ge 0$ such that $k+l=2$,

$$\|D_x^k D_t^l u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,0}-\omega} \tag{6.9b}$$

with $\omega = 4/(4(N-2)\alpha) - \varepsilon, 0 < \varepsilon \ll 1$, given by (3.7c).

Proof of Proposition 6.3. We apply Lemma 2.1 for the equation (5.10). Since we see from Proposition 6.2 and Remark 3.1 that

$$\|u(t)\|_{N\alpha}^{2\alpha} \le C\,\|u(t)\|_{q_*}^{2\alpha(1-\sigma)}\|D_x u(t)\|^{2\alpha\sigma} \le C_{[2]}(1+t)^{-\alpha\sigma} \tag{6.10}$$

with $0 < \sigma \le 1$, we have that

$$\|Df(u(t))\|^2 \le C\,\|u(t)\|_{N\alpha}^{2\alpha}E_2(t) \le C_{[2]}(1+t)^{-2b_0}E_2(t) \tag{6.11}$$

with $2b_0 = \alpha\sigma > 0$ ($D = D_x$ or D_t, see (5.11)). While, we see from Remark 3.1 that

$$\|Du(t)\| \leq C_{[1]}(1+t)^{-\omega_{1,0}}\,. \tag{6.12}$$

Applying Lemma 2.1 to (5.10), we obtain from (6.11) and (6.12) that

(6.13a)

$$E_2(t) \equiv \|D_x Du(t)\|^2 + \|D_t Du(t)\|^2 \leq C_{[2]}(1+t)^{-2\theta_2}\left(\leq C_{[2]}(1+t)^{-2\theta_2^{(1)}}\right),$$

where $\theta_2 = \min\{1/2 + \omega_{1,0}, \omega_{1,0} + b_0\} \geq \omega_{1,0}$ ($\equiv \theta_2^{(1)}$), i.e.,

$$\|D_x^2 u(t)\| \leq C_{[2]}(1+t)^{-\theta_2^{(1)}}\,. \tag{6.13b}$$

Then, we see from (6.10) and (6.13) that

$$\|u(t)\|_{N\alpha}^{2\alpha} \leq C\,\|D_x u(t)\|^{2\alpha(1-\xi)}\|D_x^2 u(t)\|^{2\alpha\xi} \leq C_{[2]}(1+t)^{-2b_1}\,, \tag{6.14a}$$

where $b_1 = \alpha(1-\xi)\omega_{1,0} + \alpha\xi\theta_2^{(1)} = \alpha\theta_2^{(1)}$, and

$$\xi = ((N-2)\alpha - 2)/2\alpha \tag{6.14b}$$

Thus, applying Lemma 2.1 again to (5.10), we obtain from (6.14) that

$$E_2(t) \leq C_{[2]}(1+t)^{-2\theta_2^{(2)}} \quad \text{or} \quad \|D_x^2 u(t)\| \leq C_{[2]}(1+t)^{-\theta_2^{(2)}}\,, \tag{6.15}$$

where $\theta_2^{(2)} = \min\{1/2 + \omega_{1,0}, \omega_{1,0} + b_1\} = \omega_{1,0} + b_1$. Then, we see again that

$$\|u(t)\|_{N\alpha}^{2\alpha} \leq C_{[2]}(1+t)^{-2b_2}\,, \tag{6.16}$$

where $b_2 = \alpha(1-\xi)\omega_{1,0} + \alpha\xi\theta_2^{(2)}$. Thus, we obtain that

$$E_2(t) \leq C_{[2]}(1+t)^{-2\theta_2^{(3)}} \quad \text{or} \quad \|D_x^2 u(t)\| \leq C_{[2]}(1+t)^{-\theta_2^{(3)}}\,, \tag{6.17}$$

where $\theta_2^{(3)} = \omega_{1,0} + b_2$. Repeating this procedure indefinitely, we obtain that

$$E_2(t) \leq C_{[2]}(1+t)^{-2\theta_2^{(m)}}\,, \quad m = 4, 5, \cdots\,, \tag{6.18a}$$

where $\theta_2^{(m)}$ is given by

(6.18b)
$$\begin{cases} \theta_2^{(m)} = \omega_{1,0} + d_{m-1}\,, \\ d_{m-1} = \alpha(1-\xi)\omega_{1,0} + \alpha\xi\theta_2^{(m-1)}\,, \end{cases}$$

or

(6.18b′)
$$\theta_2^{(m)} = \alpha\xi\theta_2^{(m-1)}\{1+\alpha(1-\xi)\}\omega_{1,0}$$

with ξ given by (6.14b), which gives the desired estimates (6.9). □

By the similar way as in previous sections, we shall improve the decay estimates for the case $(k,l) = (1,1),(0,2)$ in Propositions 6.1 and 6.3.

PROPOSITION 7.4. *Under the assumption of Remark 3.3 with $m = 1$, the solution of $u(t)$ of the equation (3.1) satisfies that*

(6.19a)
$$\|D_t u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,0}-\tilde{\omega}}$$

and

(6.19b)
$$\|D_x D_t u(t)\| + \|D_t^2 u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,0}-\omega-\tilde{\omega}}\,,$$

where $\tilde{\omega}$ is given by (3.7d).

Proof of Proposition 7.4. By use the equation (3.1), we see

(6.20)
$$\|D_t u(t)\| \le \|D_x^2 u(t)\| + \|D_t^2 u(t)\| + \|f(u(t))\|$$

Here, by Proposition 6.1 and (6.3) we have that

(6.21)
$$\|D_x^2 u(t)\| + \|D_t^2 u(t)\| \le C_{[2]}(1+t)^{-\omega_{1,0}-\omega}\,.$$

Also, if $\alpha \le 2/(N-2)$ we have from HYP. 1 that

(6.22a)
$$\begin{aligned} \|f(u(t))\| &\le C\,\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \le C\,\|u(t)\|^{(2-(N-2)\alpha)/2}\|D_x u(t)\|^{N\alpha/2} \\ &\le C_{[1]}(1+t)^{-\omega_{1,0}-[N\alpha-2]^+/4}\,, \end{aligned}$$

noting Remark 3.1. While, if $\alpha \geq 2/(N-2)$ we have that

$$\|f(u(t))\| \leq C\,\|D_x u(t)\|^{(4-(N-4)\alpha)/2}\|D_x^2 u(t)\|^{((N-2)\alpha)-2)/2}$$
$$\leq C_{[2]}(1+t)^{-\omega_{1,0}-\omega}. \tag{6.22b}$$

Thus, we obtain from (6.20), (6.21), and (6.22) that

$$\|D_t u(t)\| \leq C_{[2]}(1+t)^{-\omega_{1,0}-\widetilde{\omega}} \tag{6.23}$$

with $\widetilde{\omega}$ given by (3.7d).

Next, by the same methods as in the proof of Proposition 5.4, we can get the desired decay estimates (6.19b), using this estimate (6.23). □

Summing up the above Propositions 6.1, 6.3, 6.4, and Remark 3.1, we arrive at Remark 3.3 with $m = 1$. And, we can get Remark 3.3 with $m = 2$ by the same way as the above. The proof of Remark 3.3 is now finished.

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