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THE MULTIPLE HURWITZ ZETA FUNCTION

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1. Introduction

Recently the theory of multiple gamma functions, which were first introduced by Barnes [WB2], [WB3], [WB4], [WB5] and others about 1900, has been revived according to the study of determinants of Laplacians [HP1], [HP2], [O], [PS], [IV], [AV]. Vignéras [FV] gives us Weierstrass canonical product forms for multiple gamma functions by using a result of Dufresnoy and Pisot [JD]. Barnes [WB5] introduces these functions through n-ple Hurwitz zeta functions. We give detailed computation for the analytic continuation of the n-ple Hurwitz zeta functions $\zeta_n(s,a)$ which is important for us to give Barnes' approach for multiple gamma functions. We can also express some special values of n-ple Hurwitz zeta functions as n-ple Bernoulli polynomials.

2. The Analytic Continuation for the n-ple Hurwitz zeta Function

In this section we give an analytic continuation for $\zeta_n(s,a)$ by the contour integral representation. First we introduce Eisenstein's theorem [RF] which gives a criterion for the convergence of a n-ple series.

THEOREM 2.1. (Eisenstein's Theorem)

$$\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}{}' (m_1^2+m_2^2+\cdots+m_n^2)^{-\mu}$$

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converges if $\mu > \frac{n}{2}$, *where* $'$ *denotes that we exclude the case* $m_1 = m_2 = \ldots = m_n = 0$.

Let $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$. The n-ple Hurwitz zeta function $\zeta_n(s,a)$ is initially defined for $\sigma > n, a > 0$ by the series

$$\zeta_n(s,a) = \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}.$$

THEOREM 2.2. *The series for* $\zeta_n(s,a)$ *converges absolutely for* $\sigma > n$. *The convergence is uniform in every half-plane* $\sigma \geq n + \delta, \delta > 0$, *so* $\zeta_n(s,a)$ *is an analytic function of* s *in the half-plane* $\sigma > n$.

Proof. Note that, for $\sigma > 0$,

$$\begin{aligned} &\sum_{k_1,k_2,\cdots,k_n=0}^{\infty}{}' (k_1 + k_2 + \cdots + k_n)^{-\sigma} \\ &= \sum_{k_1,k_2,\cdots,k_n=0}^{\infty}{}' [(k_1 + k_2 + \cdots + k_n)^2]^{-\frac{\sigma}{2}} \\ &\leq \sum_{k_1,k_2,\cdots,k_n=0}^{\infty}{}' (k_1^2 + k_2^2 + \cdots + k_n^2)^{-\frac{\sigma}{2}}, \end{aligned}$$

in which the last series is convergent for $\sigma > n$ by Eisenstein's theorem. Thus all statements in Theorem 2.2 follow from the inequalities

$$\begin{aligned} &\sum_{k_1,k_2,\cdots,k_n=0}^{\infty} |(a + k_1 + k_2 + \cdots + k_n)^{-s}| \\ &= \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-\sigma} \\ &\leq \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-n-\delta}. \end{aligned}$$

THEOREM 2.3. *For $\sigma > n$ we have the integral representation*

$$\Gamma(s)\zeta_n(s,a) = \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-e^{-x})^n}dx.$$

Proof. Note that, for $\sigma > 0$,

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$$

First we keep s real, $s > 1$, and then extend the result to complex s by analytic continuation. In the integral for $\Gamma(s)$ we make the change of the variable $x = (a + k_1 + k_2 + \cdots + k_n)t$, *where* $k_i = 0, 1, 2, \ldots, 1 \leq i \leq n$, to obtain

$$\Gamma(s) = (a + k_1 + k_2 + \cdots + k_n)^s \int_0^\infty e^{-(a+k_1+k_2+\cdots+k_n)t}t^{s-1}dt$$

or

$$(a + k_1 + k_2 + \cdots + k_n)^{-s}\Gamma(s) = \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t}e^{-at}t^{s-1}dt.$$

Summing over all $k_i \geq 0, 1 \leq i \leq n$, we find

$$\zeta_n(s,a)\Gamma(s) = \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t}e^{-at}t^{s-1}dt,$$

the series on the right being convergent if $s > n$.

Now we wish to interchange the sum and integral. The simplest way to justify this is to regard the integrand as a Lebesgue integral. Since the integrand is nonnegative, Levi's convergence theorem (Theorem 10.25 in [TM]) tells us that the series

$$\sum_{k_1,k_2,\cdots,k_n=0}^{\infty} \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t}e^{-at}t^{s-1}dt,$$

converges almost everywhere to a sum function which is Lebesgue-integrable on $[0,+\infty)$ and that

$$\begin{aligned}
&\zeta_n(s,a)\Gamma(s) \\
&= \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} \int_0^{\infty} e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} dt \\
&= \int_0^{\infty} \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} dt
\end{aligned}$$

But if $t > 0$ we have $0 < e^{-t} < 1$ and hence

$$\sum_{k=0}^{\infty} e^{-kt} = \frac{1}{1-e^{-t}},$$

the series being a geometric series. Therefore we have

$$\sum_{k_1,k_2,\cdots,k_n=0}^{\infty} e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} = \frac{e^{-at} t^{s-1}}{(1-e^{-t})^n}$$

almost everywhere on $[0,+\infty)$, in fact everywhere except at 0, so

$$\begin{aligned}
&\zeta_n(s,a)\Gamma(s) \\
&= \int_0^{\infty} \sum_{k_1,k_2,\cdots,k_n=0}^{\infty} e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} dt \\
&= \int_0^{\infty} \frac{e^{-at} t^{s-1}}{(1-e^{-t})^n} dt.
\end{aligned}$$

This proves (2.1) for real $s > n$. To extend it all complex $s = \sigma + it$ with $\sigma > n$ we note that both members in the left side of (2.1) are analytic for $\sigma > n$. To show that the right member is analytic we assume $n + \delta \le \sigma \le c$, where $c > n$ and $\delta > 0$ and write

$$\begin{aligned}
\int_0^{\infty} \left| \frac{e^{-at} t^{s-1}}{(1-e^{-t})^n} \right| dt &= \int_0^{\infty} \frac{e^{-at} t^{\sigma-1}}{(1-e^{-t})^n} dt \\
&= \left(\int_0^1 + \int_1^{\infty} \right) \frac{e^{-at} t^{\sigma-1}}{(1-e^{-t})^n} dt.
\end{aligned}$$

If $0 \le t \le 1$ we have $t^{\sigma-n} \le t^{\delta}$, and if $t \ge 1$ we have $t^{\sigma-n} \le t^{c-n}$. Also since $e^t - 1 \ge t$ for $t \ge 0$ we have

$$\begin{aligned}
&\int_0^1 \frac{e^{-at}t^{\sigma-1}}{(1-e^{-t})^n} dt \\
&\le \int_0^1 \frac{e^{(n-a)t}t^{\delta+n-1}}{(e^t-1)^n} dt \\
&\le \begin{cases} e^{(n-a)} \int_0^1 t^{\delta-1}\, dt = \frac{e^{(n-a)}}{\delta} & \text{if } 0 < a \le n, \\ \int_0^1 t^{\delta-1}\, dt = \frac{1}{\delta} & \text{if } a > n. \end{cases}
\end{aligned}$$

and

$$\int_1^\infty \frac{e^{-at}t^{\sigma-1}}{(1-e^{-t})^n} \le \int_0^\infty \frac{e^{-at}t^{c-1}}{(1-e^{-t})^n} dt = \Gamma(c)\zeta_n(c,a).$$

This shows that the integral in (2.1) converges uniformly in every strip $n + \delta \le \sigma \le c, where\ \delta > 0$, and therefore represents an analytic function in every such strip, hence also in the half-plane $\sigma > n$. Therefore, by analytic continuation, (2.1) holds for all s with $\sigma > n$.

To extend $\zeta_n(s,a)$ beyond the line $\sigma = n$ we derive another representation in terms of a contour integral. The contour C is a loop around the positive real axis, as shown in Fig.

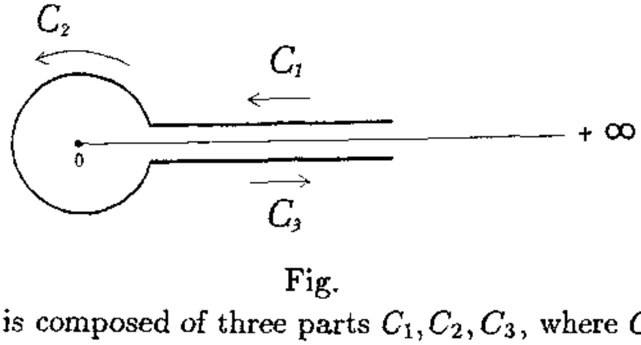

Fig.

The loop is composed of three parts C_1, C_2, C_3, where C_2 is a positively oriented circle of radius $c < 2\pi$ about the origin, and C_1, C_3 are the upper and lower edges of a cut in the z- plane along the positive real axis, traversed as shown in Fig. This means that we use the parametrizations $-z = re^{-\pi i}$ on C_1 and $-z = re^{\pi i}$ on C_3, where r varies from c to $+\infty$.

THEOREM 2.4. *If $a > 0$, the function defined by the contour integral*

$$I_n(s,a) = -\frac{1}{2\pi i}\int_C \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^n}dz$$

is an entire function of s. Moreover, we have

$$\zeta_n(s,a) = \Gamma(1-s)I_n(s,a) \ if \ \sigma > n.$$

Proof. Here $(-z)^s$ means $r^s e^{-\pi i s}$ on C_1 and $r^s e^{\pi i s}$ on C_3. We consider an arbitrary compact disk $|s| \le M$ and prove that the integrals along C_1 and C_3 converge uniformly on every such disk. Since the integrand is an entire function of s this will prove that $I_n(s,a)$ is entire. Along C_1 we have, for $r \ge 1$,

$$|(-z)^{s-1}| = r^{\sigma-1}|e^{-\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{\pi t} \le r^{M-1}e^{\pi M}$$

since $|s| \le M$. Similarly, along C_3 we have, for $r \ge 1$,

$$|(-z)^{s-1}| = r^{\sigma-1}|e^{\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{-\pi t} \le r^{M-1}e^{\pi M}.$$

Hence on either C_1 or C_3 we have, for $r \ge 1$,

$$\left|\frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^n}\right| \le \frac{r^{M-1}e^{\pi M}e^{-ar}}{(1-e^{-r})^n} = \frac{r^{M-1}e^{\pi M}e^{(n-a)r}}{(e^r-1)^n}.$$

But $\int_c^\infty r^{M-1}e^{-ar}dr$ converges if $c > 0$ this shows that the integrals along C_1 and C_3 converge uniformly on every compact disk $|s| \le M$, and hence $I_n(s,a)$ is an entire function of s.

To prove (2.2) we write

$$-2\pi i I_n(s,a) = (\int_{C_1} + \int_{C_2} + \int_{C_3})(-z)^{s-1}g(-z)dz$$

where $g(-z) = e^{-az}/(1-e^{-z})^n$. On C_1 and C_3 we have $g(-z) = g(-r)$, and on C_2 we write $-z = ce^{i\theta}$ where θ varies from 2π to 0. This gives us

$$\begin{aligned}
-2\pi i I_n(s,a) &= \int_\infty^c r^{s-1} e^{-\pi i(s-1)} g(-r)dr \\
&\quad - i\int_{2\pi}^0 c^{s-1} e^{(s-1)i\theta} c e^{i\theta} g(ce^{i\theta})d\theta \\
&\quad + \int_c^\infty r^{s-1} e^{\pi i(s-1)} g(-r)dr \\
&= -2i\sin(\pi s)\int_c^\infty r^{s-1} g(-r)dr \\
&\quad - ic^s\int_{2\pi}^0 e^{is\theta} g(ce^{i\theta})d\theta.
\end{aligned}$$

Dividing by $-2i$, we get

$$\pi I_n(s,a) = \sin(\pi s) I_1(s,c) + I_2(s,c),$$

where

$$\begin{aligned}
I_1(s,c) &= \int_c^\infty r^{s-1} g(-r)dr \\
I_2(s,c) &= \frac{c^s}{2}\int_{2\pi}^0 e^{is\theta} g(ce^{i\theta})d\theta.
\end{aligned}$$

Now let $c \to 0$. We can find

$$\lim_{c\to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1}e^{-ar}}{(1-e^{-r})^n}dr = \Gamma(s)\zeta_n(s,a),$$

if $\sigma > n$. We will show next that $\lim_{c\to 0} I_2(s,c) = 0$. To do this note that $g(-z)$ is analytic in $|z| < 2\pi$ except for a pole of order n at $z = 0$. Therefore $z^n g(-z)$ is analytic everywhere inside $|z| < 2\pi$ and hence is bounded there, say $|g(-z)| \le A/|z|^n$, where $|z| = c < 2\pi$ and A is a constant. Therefore we have

$$|I_2(s,c)| \le \frac{c^\sigma}{2}\int_0^{2\pi} e^{-t\theta}\frac{A}{c^n}d\theta \le \pi A e^{2\pi|t|} e^{\sigma-n}.$$

If $\sigma > n$ and $c \to 0$ we can find $I_2(s,c) \to 0$ hence

$$\pi I_n(s,a) = \sin(\pi s)\Gamma(s)\zeta_n(s,a).$$

Since $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$ this proves (2.2).

In the equation $\zeta_n(s,a) = \Gamma(1-s)I_n(s,a)$, valid for $\sigma > n$, the function $I_n(s,a)$ and $\Gamma(1-s)$ are meaningful for every complex s. Therefore we can use this equation to define $\zeta_n(s,a)$ for $\sigma \le n$.

DEFINITION 2.5. *If $\sigma \le n$ we define $\zeta_n(s,a)$ by the equation*

$$\zeta_n(s,a) = \Gamma(1-s)I_n(s,a).$$

This equation provides the analytic continuation of $\zeta_n(s,a)$ in the entire s-plane.

THEOREM 2.6. *The function $\zeta_n(s,a)$ so defined is analytic for all s except for simple poles at $s = l, 1 \le l \le n$, with their residues*

$$\frac{1}{(n-l)!(l-1)!}\lim_{z\to 0}\frac{d^{n-l}}{dz^{n-l}}\frac{z^n e^{-az}}{(1-e^{-z})^n}.$$

In particular, when $s = n$, its residue is $1/(n-1)!$.

Proof. Since $I_n(s,a)$ is entire, the only possible singularities of $\zeta_n(s,a)$ are the poles of $\Gamma(1-s)$. Since $1/\Gamma(1-s)$ has simple zeros at $s = 1,2,3,\cdots, \Gamma(1-s)$ has simple poles at $s = 1,2,3,\cdots$. But Theorem 2.4 shows that $\zeta_n(s,a)$ is analytic at $s = n+1, n+2, \cdots$, so $s = 1,2,3,\cdots,n$ are the only poles of $\zeta_n(s,a)$.

Now we show that there are poles at $s = l, 1 \le l \le n$, with their residues

$$\frac{1}{(n-l)!(l-1)!}\lim_{z\to 0}\frac{d^{n-l}}{dz^{n-l}}\frac{z^n e^{-az}}{(1-e^{-z})^n}.$$

If s is any integer, say $s = l$, the integrand in the contour integral for $I_n(s,a)$ takes the same values on C_1 as on C_3 and hence the integral along C_1 and C_3 cancel, leaving

$$\begin{aligned} I_n(l,a) &= -\frac{1}{2\pi i}\int_{C_2}\frac{(-z)^{l-1}e^{-az}}{(1-e^{-z})^n}dz \\ &= -\mathrm{Res}_{z=0}\frac{(-z)^{l-1}e^{-az}}{(1-e^{-z})^n}. \end{aligned}$$

We can show that $(-z)^{l-1}e^{-az}/(1-e^{-z})^n$ has a pole of order $n+1-l$ at $z=0, 1 \leq l \leq n$. Therefore we have

$$I_n(l,a) = \frac{(-1)^l}{(n-l)!} \lim_{z\to 0} \frac{d^{n-l}}{dz^{n-l}} \frac{z^n e^{-az}}{(1-e^{-z})^n}.$$

To find the residue of $\zeta_n(s,a)$ at $s=l, 1 \leq l \leq n$, we compute the limit

$$\begin{aligned}
\lim_{s\to l}(s-l)\zeta_n(s,a) &= \lim_{s\to l}(s-l)\Gamma(1-s)I_n(s,a) \\
&= I_n(l,a)\lim_{s\to l}(s-l)\Gamma(1-s) \\
&= I_n(l,a)\lim_{s\to l}(s-l)\frac{\pi}{\Gamma(s)\sin\pi s} \\
&= \frac{\pi I_n(l,a)}{\Gamma(l)}\lim_{s\to l}\frac{s-l}{\sin\pi s} \\
&= \frac{I_n(l,a)}{\Gamma(l)}\frac{1}{\cos(\pi l)} \\
&= \frac{I_n(l,a)}{(-1)^l(l-1)!} \\
&= \frac{1}{(n-l)!(l-1)!}\lim_{z\to 0}\frac{d^{n-l}}{dz^{n-l}}\frac{z^n e^{-az}}{(1-e^{-z})^n}.
\end{aligned}$$

In particular, the residue of $\zeta_n(s,a)$ at $s=n$ is $1/(n-1)!$.

The generalized zeta function (or Hurwitz zeta function) $\zeta(s,a)$ is defind for $\sigma > 1, a > 0$ by the series

$$\zeta(s,a) = \sum_{k=0}^{\infty}(a+k)^{-s}.$$

In particular, when $a=1, \zeta(s,1) = \sum_{k=1}^{\infty} k^{-s}$ is usually called the Riemann zeta function, denoted by $\zeta(s)$ [TW]. Corollary 2.7 follows easily from Theorem 2.6.

COROLLARY 2.7. *$\zeta(s,a)$ can be continued analytically to the entire s-plane except for a simple pole only at $s=1$ with its residue 1.*

3. Some Special Values of $\zeta_n(s,a)$

Now the value of $\zeta_n(-l,a)$ can be calculated explicitly if l is a non-negative integer. Taking $s=-l$ in the relation $\zeta_n(s,a)=\Gamma(1-s)I_n(s,a)$ we can find

$$\zeta_n(-l,a)=\Gamma(1+l)I_n(-l,a)=l!I_n(-l,a).$$

We also have

$$\begin{aligned} I_n(-l,a) &= -\frac{1}{2\pi i}\int_{C_2}\frac{(-z)^{-l-1}e^{-az}}{(1-e^{-z})^n}dz \\ &= -\mathrm{Res}_{z=0}\frac{(-z)^{-l-1}e^{-az}}{(1-e^{-z})^n}. \end{aligned}$$

The calculation of this residue leads to an interesting class of functions known as Bernoulli polynomials.

DEFINITION 3.1. [HB]. *For any complex z we define the functions $B_l(x)$ by the equation*

$$\frac{ze^{xz}}{e^z-1}=\sum_{l=0}^{\infty}\frac{B_l(x)}{l!}z^l,\ \textit{where}\ |z|<2\pi.$$

The functions $B_l(x)$ are called l-th Bernoulli polynomials and the numbers $B_l(0)$ are called Bernoulli numbers and are denoted by B_l. Thus

,

$$\frac{z}{e^z-1}=\sum_{l=0}^{\infty}\frac{B_l}{l!}z^l,\ \textit{where}\ |z|<2\pi.$$

The Bernoulli polynomials and numbers of order n are defined respectively by, for any complex number x,

$$\frac{z^n e^{xz}}{(e^z-1)^n} = \sum_{l=0}^{\infty} B_l^{(n)}(x)\frac{z^l}{l!}, \text{ where } |z| < 2\pi,$$

$$\frac{z^n}{(e^z-1)^n} = \sum_{l=0}^{\infty} B_l^{(n)}\frac{z^l}{l!}, \text{ where } |z| < 2\pi.$$

Note that $B_l^{(1)}(x) = B_l(x), B_l^{(1)}(0) = B_l$ *and* $B_l^{(n)}(0) = B_l^{(n)}$.

There are lots of formulas involved in Bernoulli polynomials. Here we give some of them:

$$B_l^{(n)}(x) = \sum_{k=0}^{l}\binom{l}{k}B_k^{(n)}x^{l-k}.$$

The Bernoulli polynomials satisfy the addition formula

$$B_l^{(n)}(x+y) = \sum_{k=0}^{l}\binom{n}{k}B_k^{(n)}(x)y^{l-k}.$$

THEOREM 3.2. The *Bernoulli polynomials* $B_l^{(n)}(x)$ *satisfy the equation*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}B_l^{(n)}(x+k) = \frac{l!}{(l-n)!}x^{l-n} \text{ if } l \geq n.$$

In particular, when $n = 1, B_l(x+1) - B_l(x) = lx^{l-1}$ *if* $l \geq 1$.

Proof. We have

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_l^{(n)}(x+k)}{l!} z^l \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{z^n}{(e^z-1)^n} e^{(x+k)z} \\
&= \left[\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^{kz} \right] \frac{z^n e^{xz}}{(e^z-1)^n} \\
&= \frac{(e^z-1)^n}{(e^z-1)^n} z^n e^{xz} \\
&= \sum_{m=0}^{\infty} \frac{x^m}{m!} z^{n+m} \\
&= \sum_{l=n}^{\infty} \frac{x^{l-n}}{(l-n)!} z^l,
\end{aligned}$$

For $l \geq n$, equating the coefficients of z^l, the theorem follows.

THEOREM 3.3. *$B_l^{(n)}(n-x) = (-1)^l B_l^{(n)}(x)$ for every integer $l \geq 0$.*

Proof. For $|z| < 2\pi$, we have

$$\frac{z e^{(n-x)z}}{(e^z-1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(n-x)}{l!} z^l.$$

Replacing z by $-z$, we have

$$\frac{(-z)^n e^{(x-n)z}}{(e^{-z}-1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(n-x)}{l!} (-z)^l.$$

On the other hand

$$\frac{(-z)^n e^{(x-n)z}}{(e^{-z}-1)^n} = \frac{z^n e^{xz}}{(e^z-1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(x)}{l!} z^l.$$

Equating coefficients of z^l, we obtain the desired results.

THEOREM 3.4. *For every integer $l \geq 0$, we have*

$$\zeta_n(-l,a) = (-1)^l \frac{l!}{(n+l)!} B_{n+l}^{(n)}(n-a).$$

Proof. As noted earlier,we have $\zeta_n(-l,a) = l! I_n(-l,a)$. Now

$$\begin{aligned} I_n(-l,a) &= -\mathrm{R}es_{z=0} \frac{(-z)^{-l-1}e^{-az}}{(1-e^{-z})^n} \\ &= (-1)^l \mathrm{R}es_{z=0} z^{-l-1} \frac{e^{(n-a)s}}{(e^z-1)^n} \\ &= (-1)^l \mathrm{R}es_{z=0} z^{-n-l-1} \frac{z^n e^{(n-a)z}}{(e^z-1)^n} \\ &= (-1)^l \mathrm{R}es_{z=0} z^{-n-l-1} \sum_{k=0}^{\infty} B_k^{(n)}(n-a) \frac{z^k}{k!} \\ &= (-1)^l \frac{B_{n+l}^{(n)}(n-a)}{(n+l)!}, \end{aligned}$$

from which we obtain (3.4).

From Theorems 3.3 and 3.4 we have the following.

CROLLARY 3.5. *For every integer $l \geq 0$ we have*

$$\zeta_n(-l,a) = (-1)^n \frac{l!}{(n+l)!} B_{n+l}^{(n)}(a).$$

In particular, $\zeta(-l,a) = -B_{l+1}(a)/(l+1)$.

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