THE MULTIPLE HURWITZ ZETA FUNCTION

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1. Introduction

Recently the theory of multiple gamma functions, which were first introduced by Barnes [WB2], [WB3], [WB4], [WB5] and others about 1900, has been revived according to the study of determinants of Laplacians [HP1], [HP2], [O], [PS], [IV], [AV]. Vigneras [FV] gives us Weierstrass canonical product forms for multiple gamma functions by using a result of Dufresnoy and Pisot [JD]. Barnes [WB5] introduces these functions through n-ple Hurwitz zeta functions. We give detailed computation for the analytic continuation of the n-ple Hurwitz zeta functions \( \zeta_n(s, a) \) which is important for us to give Barnes' approach for multiple gamma functions. We can also express some special values of n-ple Hurwitz zeta functions as n-ple Bernoulli polynomials.

2. The Analytic Continuation for the n-ple Hurwitz zeta Function

In this section we give an analytic continuation for \( \zeta_n(s, a) \) by the contour integral representation. First we introduce Eisenstein’s theorem [RF] which gives a criterion for the convergence of a n-ple series.

**THEOREM 2.1. (Eisenstein’s Theorem)**

\[
\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \left( m_1^2 + m_2^2 + \cdots + m_n^2 \right)^{-\mu}
\]

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converges if \( \mu > \frac{n}{2} \), where ' denotes that we exclude the case \( m_1 = m_2 = \ldots = m_n = 0 \).

Let \( s = \sigma + it \), where \( \sigma, t \in \mathbb{R} \). The \( n \)-ple Hurwitz zeta function \( \zeta_n(s, a) \) is initially defined for \( \sigma > n, a > 0 \) by the series

\[
\zeta_n(s, a) = \sum_{k_1, k_2, \ldots, k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}.
\]

**THEOREM 2.2.** The series for \( \zeta_n(s, a) \) converges absolutely for \( \sigma > n \). The convergence is uniform in every half-plane \( \sigma > n + \delta, \delta > 0 \), so \( \zeta_n(s, a) \) is an analytic function of \( s \) in the half-plane \( \sigma > n \).

**Proof.** Note that, for \( \sigma > 0 \),

\[
\sum_{k_1, k_2, \ldots, k_n=0}^{\infty} (k_1 + k_2 + \cdots + k_n)^{-\sigma} = \sum_{k_1, k_2, \ldots, k_n=0}^{\infty} [(k_1 + k_2 + \cdots + k_n)^2]^{-\sigma}
\]

\[
\leq \sum_{k_1, k_2, \ldots, k_n=0}^{\infty} (k_1^2 + k_2^2 + \cdots + k_n^2)^{-\frac{\sigma}{2}},
\]

in which the last series is convergent for \( \sigma > n \) by Eisenstein's theorem. Thus all statements in Theorem 2.2 follow from the inequalities

\[
\sum_{k_1, k_2, \ldots, k_n=0}^{\infty} |(a + k_1 + k_2 + \cdots + k_n)^{-\delta}|
\]

\[
= \sum_{k_1, k_2, \ldots, k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-\sigma}
\]

\[
\leq \sum_{k_1, k_2, \ldots, k_n=0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-n-\delta}.
\]
The Multiple Hurwitz zeta Function

**Theorem 2.3.** For \( \sigma > n \) we have the integral representation

\[
\Gamma(s) \zeta_n(s, a) = \int_0^\infty \frac{x^{s-1} e^{-ax}}{(1 - e^{-x})^n} \, dx.
\]

**Proof.** Note that, for \( \sigma > 0 \),

\[
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx
\]

First we keep \( s \) real, \( s > 1 \), and then extend the result to complex \( s \) by analytic continuation. In the integral for \( \Gamma(s) \) we make the change of the variable

\[
x = (a + k_1 + k_2 + \cdots + k_n)t,
\]

where \( k_1, k_2, \ldots, 1 \leq i \leq n \), to obtain

\[
\Gamma(s) = (a + k_1 + k_2 + \cdots + k_n)^s \int_0^\infty e^{-(a+k_1+k_2+\cdots+k_n)t} t^{s-1} \, dt
\]

or

\[
(a + k_1 + k_2 + \cdots + k_n)^{-s} \Gamma(s) = \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} \, dt.
\]

Summing over all \( k_i \geq 0, 1 \leq i \leq n \), we find

\[
\zeta_n(s, a) \Gamma(s) = \sum_{k_1, k_2, \ldots, k_n=0}^\infty \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} \, dt,
\]

the series on the right being convergent if \( s > n \).

Now we wish to interchange the sum and integral. The simplest way to justify this is to regard the integrand as a Lebesgue integral. Since the integrand is nonnegative, Levi's convergence theorem (Theorem 10.25 in [TM]) tells us that the series

\[
\sum_{k_1, k_2, \ldots, k_n=0}^\infty \int_0^\infty e^{-(k_1+k_2+\cdots+k_n)t} e^{-at} t^{s-1} \, dt,
\]
converges almost everywhere to a sum function which is Lebesgue-integrable on \([0, +\infty)\) and that

\[
\zeta_n(s, a) \Gamma(s) = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} \int_0^\infty e^{-(k_1 + k_2 + \cdots + k_n)t} e^{-at^s} \, dt = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} e^{-(k_1 + k_2 + \cdots + k_n)t} e^{-at^s} \, dt.
\]

But if \(t > 0\) we have \(0 < e^{-t} < 1\) and hence

\[
\sum_{k=0}^{\infty} e^{-kt} = \frac{1}{1 - e^{-t}},
\]

the series being a geometric series. Therefore we have

\[
\sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} e^{-(k_1 + k_2 + \cdots + k_n)t} e^{-at^s} \, dt = \frac{e^{-at^s} - 1}{(1 - e^{-t})^n}
\]

almost everywhere on \([0, +\infty)\), in fact everywhere except at 0, so

\[
\zeta_n(s, a) \Gamma(s) = \int_0^\infty \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} e^{-(k_1 + k_2 + \cdots + k_n)t} e^{-at^s} \, dt = \int_0^\infty \frac{e^{-at^s} - 1}{(1 - e^{-t})^n} \, dt.
\]

This proves (2.1) for real \(s > n\). To extend it all complex \(s = \sigma + it\) with \(\sigma > n\) we note that both members in the left side of (2.1) are analytic for \(\sigma > n\). To show that the right member is analytic we assume \(n + \delta \leq \sigma \leq c\), where \(c > n\) and \(\delta > 0\) and write

\[
\int_0^\infty \frac{e^{-at^s} - 1}{(1 - e^{-t})^n} \, dt = \int_0^\infty \frac{e^{-at^\sigma} - 1}{(1 - e^{-t})^n} \, dt = \left( \int_0^1 + \int_1^\infty \right) \frac{e^{-at^\sigma} - 1}{(1 - e^{-t})^n} \, dt.
\]
If \( 0 \leq t \leq 1 \) we have \( t^{\sigma-n} \leq t^\delta \), and if \( t \geq 1 \) we have \( t^{\sigma-n} \leq t^{e-n} \). Also since \( e^t - 1 \geq t \) for \( t \geq 0 \) we have

\[
\int_0^1 \frac{e^{-at}t^{\sigma-1}}{(1-e^{-t})^n} dt = \int_0^1 \frac{e^{(n-a)t_0^{\delta+n-1}}}{(e^t - 1)^n} dt = \left\{
\begin{align*}
& \frac{e^{(n-a)}}{\delta} t^{\delta-1} dt = \frac{e^{(n-a)}}{\delta} \quad \text{if } 0 < a \leq n, \\
& \int_0^1 t^{\delta-1} dt = \frac{1}{\delta} \quad \text{if } a > n.
\end{align*}
\right.
\]

and

\[
\int_1^\infty \frac{e^{-at}t^{\sigma-1}}{(1-e^{-t})^n} dt \leq \int_0^\infty \frac{e^{-at}t^{\sigma-1}}{(1-e^{-t})^n} dt = \Gamma(c)\zeta_n(c, a).
\]

This shows that the integral in (2.1) converges uniformly in every strip \( n + \delta \leq \sigma \leq c \), where \( \delta > 0 \), and therefore represents an analytic function in every such strip, hence also in the half-plane \( \sigma > n \). Therefore, by analytic continuation, (2.1) holds for all \( s \) with \( \sigma > n \).

To extend \( \zeta_n(s, a) \) beyond the line \( \sigma = n \) we derive another representation in terms of a contour integral. The contour \( C \) is a loop around the positive real axis, as shown in Fig.

To extend \( \zeta_n(s, a) \) beyond the line \( \sigma = n \) we derive another representation in terms of a contour integral. The contour \( C \) is a loop around the positive real axis, as shown in Fig.

The loop is composed of three parts \( C_1, C_2, C_3 \), where \( C_2 \) is a positively oriented circle of radius \( c < 2\pi \) about the origin, and \( C_1, C_3 \) are the upper and lower edges of a cut in the \( z \)-plane along the positive real axis, traversed as shown in Fig. This means that we use the parametrizations \( -z = re^{-\pi i} \) on \( C_1 \) and \( -z = re^{\pi i} \) on \( C_3 \), where \( r \) varies from \( c \) to \( +\infty \).
THEOREM 2.4. If $a > 0$, the function defined by the contour integral

$$I_n(s, a) = -\frac{1}{2\pi i} \int_C \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^n} dz$$

is an entire function of $s$. Moreover, we have

$$\zeta_n(s, a) = \Gamma(1-s)I_n(s, a) \text{ if } \sigma > n.$$ 

Proof. Here $(-z)^s$ means $r^se^{-\pi is}$ on $C_1$ and $r^se^{\pi is}$ on $C_3$. We consider an arbitrary compact disk $|s| \leq M$ and prove that the integrals along $C_1$ and $C_3$ converge uniformly on every such disk. Since the integrand is an entire function of $s$ this will prove that $I_n(s, a)$ is entire. Along $C_1$ we have, for $r \geq 1$,

$$|(-z)^{s-1}| = r^{\sigma-1}|e^{-\pi i(s-1+it)}| = r^{\sigma-1}e^{\pi t} \leq r^{M-1}e^{\pi M}$$

since $|s| \leq M$. Similarly, along $C_3$ we have, for $r \geq 1$,

$$|(-z)^{s-1}| = r^{\sigma-1}|e^{\pi i(s-1+it)}| = r^{\sigma-1}e^{-\pi t} \leq r^{M-1}e^{\pi M}.$$ 

Hence on either $C_1$ or $C_3$ we have, for $r \geq 1$,

$$\left|\frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^n}\right| \leq \frac{r^{M-1}e^{\pi M}e^{-ar}}{(1-e^{-r})^n} = \frac{r^{M-1}e^{\pi M}e^{(n-a)r}}{(e^r-1)^n}.$$ 

But $\int_0^\infty r^{M-1}e^{-ar}dr$ converges if $c > 0$ this shows that the integrals along $C_1$ and $C_3$ converge uniformly on every compact disk $|s| \leq M$, and hence $I_n(s, a)$ is an entire function of $s$.

To prove (2.2) we write

$$-2\pi i I_n(s, a) = (\int_{C_1} + \int_{C_2} + \int_{C_3})(-z)^{s-1}g(-z)dz$$

where $g(-z) = e^{-az}/(1-e^{-z})^n$. On $C_1$ and $C_3$ we have $g(-z) = g(-r)$, and on $C_2$ we write $-z = ce^{i\theta}$ where $\theta$ varies from $2\pi$ to $0$. This gives
us
\[-2\pi i I_n(s, a) = \int_\infty^c r^{s-1} e^{-\pi i (s-1)} g(-r) dr\]
\[-i \int_0^\infty \frac{r^{s-1} e^{(s-1) i \theta} e^{i \theta} g(c e^{i \theta}) d\theta}{2\pi}
+ \int_c^\infty r^{s-1} e^{\pi i (s-1)} g(-r) dr\]
\[= -2i \sin(\pi s) \int_c^\infty r^{s-1} g(-r) dr\]
\[-ic^s \int_0^\infty e^{i s \theta} g(c e^{i \theta}) d\theta.\]

Dividing by $-2i$, we get
\[\pi I_n(s, a) = \sin(\pi s) I_1(s, c) + I_2(s, c),\]

where
\[I_1(s, c) = \int_c^\infty r^{s-1} g(-r) dr\]
\[I_2(s, c) = \frac{c^s}{2} \int_0^\infty e^{i s \theta} g(c e^{i \theta}) d\theta.\]

Now let $c \to 0$. We can find
\[\lim_{c \to 0} I_1(s, c) = \int_0^\infty \frac{r^{s-1} e^{-ar}}{(1 - e^{-r})^n} dr = \Gamma(s) \zeta_n(s, a),\]

if $\sigma > n$. We will show next that $\lim_{c \to 0} I_2(s, c) = 0$. To do this note that $g(-z)$ is analytic in $|z| < 2\pi$ except for a pole of order $n$ at $z = 0$. Therefore $z^ng(-z)$ is analytic everywhere inside $|z| < 2\pi$ and hence is bounded there, say $|g(-z)| \leq A/|z|^n$, where $|z| = c < 2\pi$ and $A$ is a constant. Therefore we have
\[|I_2(s, c)| \leq \frac{c^\sigma}{2} \int_0^{2\pi} e^{-t\theta} A e^{\frac{1}{c^n}} d\theta \leq \pi A e^{2\pi |\sigma-n|}.\]
If $\sigma > n$ and $c \to 0$ we can find $I_2(s, c) \to 0$ hence
\[
\pi I_n(s, a) = \sin(\pi s) \Gamma(s) \zeta_n(s, a).
\]
Since $\Gamma(s) \Gamma(1 - s) = \pi / \sin \pi s$ this proves (2.2).

In the equation $\zeta_n(s, a) = \Gamma(1 - s) I_n(s, a)$, valid for $\sigma > n$, the function $I_n(s, a)$ and $\Gamma(1 - s)$ are meaningful for every complex $s$. Therefore we can use this equation to define $\zeta_n(s, a)$ for $\sigma \leq n$.

**DEFINITION 2.5.** If $\sigma \leq n$ we define $\zeta_n(s, a)$ by the equation
\[
\zeta_n(s, a) = \Gamma(1 - s) I_n(s, a).
\]

This equation provides the analytic continuation of $\zeta_n(s, a)$ in the entire $s$-plane.

**THEOREM 2.6.** The function $\zeta_n(s, a)$ so defined is analytic for all $s$ except for simple poles at $s = l, 1 \leq l \leq n$, with their residues
\[
\frac{1}{(n - l)!(l - 1)!} \lim_{z \to 0} \frac{d^{n-l} z^n e^{-az}}{dz^{n-l} (1 - e^{-z})^n}.
\]
In particular, when $s = n$, its residue is $1/(n - 1)!$.

**Proof.** Since $I_n(s, a)$ is entire, the only possible singularities of $\zeta_n(s, a)$ are the poles of $\Gamma(1 - s)$. Since $1/\Gamma(1 - s)$ has simple zeros at $s = 1, 2, 3, \ldots, \Gamma(1 - s)$ has simple poles at $s = 1, 2, 3, \ldots$. But Theorem 2.4 shows that $\zeta_n(s, a)$ is analytic at $s = n + 1, n + 2, \ldots$, so $s = 1, 2, 3, \ldots, n$ are the only poles of $\zeta_n(s, a)$.

Now we show that there are poles at $s = l, 1 \leq l \leq n$, with their residues
\[
\frac{1}{(n - l)!(l - 1)!} \lim_{z \to 0} \frac{d^{n-l} z^n e^{-az}}{dz^{n-l} (1 - e^{-z})^n}.
\]
If $s$ is any integer, say $s = l$, the integrand in the contour integral for $I_n(s, a)$ takes the same values on $C_1$ as on $C_3$ and hence the integral along $C_1$ and $C_3$ cancel, leaving
\[
I_n(l, a) = -\frac{1}{2\pi i} \int_{C_2} \frac{(-z)^{l-1} e^{-az}}{(1 - e^{-z})^n} dz
\]
\[
= -\text{Res}_{z=0} \frac{(-z)^{l-1} e^{-az}}{(1 - e^{-z})^n}.
\]
We can show that \((-z)^{l-1} e^{-az}/(1 - e^{-z})^n\) has a pole of order \(n + 1 - l\) at \(z = 0, 1 \leq l \leq n\). Therefore we have

\[
I_n(l, a) = \frac{(-1)^l}{(n-l)!} \lim_{z \to 0} \frac{d^{n-l}}{dz^{n-l}} \frac{z^n e^{-az}}{(1 - e^{-z})^n}.
\]

To find the residue of \(\zeta_n(s, a)\) at \(s = l, 1 \leq l \leq n\), we compute the limit

\[
\lim_{s \to l} (s - l) \zeta_n(s, a) = \lim_{s \to l} (s - l) \Gamma(1 - s) I_n(s, a)
\]

\[
= I_n(l, a) \lim_{s \to l} (s - l) \Gamma(1 - s)
\]

\[
= I_n(l, a) \lim_{s \to l} \frac{\pi}{\Gamma(s) \sin \pi s}
\]

\[
= \pi I_n(l, a) \lim_{s \to l} \frac{s - l}{\sin \pi s} \Gamma(l) \cos(\pi l)
\]

\[
= \frac{I_n(l, a)}{(n-l)! \Gamma(l) \cos(l)}
\]

\[
= \frac{(-1)^l}{(l-1)!} \frac{1}{\Gamma(l)} \lim_{z \to 0} \frac{d^{n-l}}{dz^{n-l}} \frac{z^n e^{-az}}{(1 - e^{-z})^n}.
\]

In particular, the residue of \(\zeta_n(s, a)\) at \(s = n\) is \(1/(n-1)!\).

The generalized zeta function (or Hurwitz zeta function) \(\zeta(s, a)\) is defined for \(\sigma > 1, a > 0\) by the series

\[
\zeta(s, a) = \sum_{k=0}^{\infty} (a + k)^{-s}.
\]

In particular, when \(a = 1, \zeta(s, 1) = \sum_{k=1}^{\infty} k^{-s}\) is usually called the Riemann zeta function, denoted by \(\zeta(s)\) [TW]. Corollary 2.7 follows easily from Theorem 2.6.
COROLLARY 2.7. $\zeta(s,a)$ can be continued analytically to the entire $s$-plane except for a simple pole only at $s = 1$ with its residue 1.

3. Some Special Values of $\zeta_n(s,a)$

Now the value of $\zeta_n(-l, a)$ can be calculated explicitly if $l$ is a non-negative integer. Taking $s = -l$ in the relation $\zeta(s,a) = \Gamma(1-s)I_n(s,a)$ we can find

$$\zeta_n(-l, a) = \Gamma(1+l)I_n(-l, a) = l!I_n(-l, a).$$

We also have

$$I_n(-l, a) = -\frac{1}{2\pi i} \int_{C_\alpha} \frac{(-z)^{-l-1}e^{-az}}{(1-e^{-z})^n} dz$$

$$= -\text{Res}_{z=0} \frac{(-z)^{-l-1}e^{-az}}{(1-e^{-z})^n}.$$

The calculation of this residue leads to an interesting class of functions known as Bernoulli polynomials.

DEFINITION 3.1. [HB]. For any complex $z$ we define the functions $B_l(x)$ by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{l=0}^{\infty} \frac{B_l(x)}{l!} z^l, \text{ where } |z| < 2\pi.$$

The functions $B_l(x)$ are called $l$-th Bernoulli polynomials and the numbers $B_l(0)$ are called Bernoulli numbers and are denoted by $B_l$. Thus

$$\frac{z}{e^z - 1} = \sum_{l=0}^{\infty} \frac{B_l}{l!} z^l, \text{ where } |z| < 2\pi.$
The Bernoulli polynomials and numbers of order $n$ are defined respectively by, for any complex number $x$,

\[
\frac{z^n e^{xz}}{(e^z - 1)^n} = \sum_{l=0}^{\infty} B_l^{(n)}(x) \frac{z^l}{l!}, \text{ where } |z| < 2\pi,
\]

\[
\frac{z^n}{(e^z - 1)^n} = \sum_{l=0}^{\infty} B_l^{(n)} \frac{z^l}{l!}, \text{ where } |z| < 2\pi.
\]

Note that $B_l^{(1)}(x) = B_l(x), B_l^{(1)}(0) = B_l$ and $B_l^{(n)}(0) = B_l^{(n)}$.

There are lots of formulas involved in Bernoulli polynomials. Here we give some of them:

\[
B_l^{(n)}(x) = \sum_{k=0}^{l} \binom{l}{k} B_k^{(n)} x^{l-k}.
\]

The Bernoulli polynomials satisfy the addition formula

\[
B_l^{(n)}(x + y) = \sum_{k=0}^{l} \binom{n}{k} B_k^{(n)}(x)y^{l-k}.
\]

**Theorem 3.2.** The Bernoulli polynomials $B_l^{(n)}(x)$ satisfy the equation

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_l^{(n)}(x + k) = \frac{l!}{(l-n)!} x^{l-n} \text{ if } l \geq n.
\]

In particular, when $n = 1, B_l(x + 1) - B_l(x) = lx^{l-1} \text{ if } l \geq 1.$
Proof. We have

\[\sum_{l=0}^{\infty} \sum_{k=0}^{n} \frac{\binom{n}{k} (-1)^{n-k} B_l^{(n)}(x+k)}{l!} z^l\]

\[= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{z^n}{(e^z - 1)^n} e^{(x+k)z}\]

\[= \left[ \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^{kz} \right] \frac{z^n e^{xz}}{(e^z - 1)^n}\]

\[= \frac{(e^z - 1)^n}{(e^z - 1)^n} z^n e^{xz}\]

\[= \sum_{m=0}^{\infty} \frac{x^m}{m!} z^{n+m}\]

\[= \sum_{l=n}^{\infty} \frac{x^{l-n}}{(l-n)!} z^l,\]

For \(l \geq n\), equating the coefficients of \(z^l\), the theorem follows.

**Theorem 3.3.** \(B_l^{(n)}(n-x) = (-1)^l B_l^{(n)}(x)\) for every integer \(l \geq 0\).

**Proof.** For \(|z| < 2\pi\), we have

\[\frac{ze^{(n-x)z}}{(e^z - 1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(n-x)}{l!} z^l.\]

Replacing \(z\) by \(-z\), we have

\[\frac{(-z)^n e^{(x-n)z}}{(e^{-z} - 1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(n-x)}{l!} (-z)^l.\]

On the other hand

\[\frac{(-z)^n e^{(x-n)z}}{(e^{-z} - 1)^n} = \frac{z^n e^{xz}}{(e^z - 1)^n} = \sum_{l=0}^{\infty} \frac{B_l^{(n)}(x)}{l!} z^l.\]

Equating coefficients of \(z^l\), we obtain the desired results.
THEOREM 3.4. For every integer \( l \geq 0 \), we have

\[
\zeta_n(-l,a) = (-1)^l \frac{l!}{(n+l)!} B_{n+l}^{(n)}(n-a).
\]

Proof. As noted earlier, we have \( \zeta_n(-l,a) = l! I_n(-l,a) \). Now

\[
I_n(-l,a) = -\text{Res}_{z=0} \frac{(z)^{-l-1}e^{-az}}{(1-e^{-z})^n}
\]

\[
= (-1)^l \text{Res}_{z=0} z^{-n-1} \frac{e^{z(n-a)}}{(e^z-1)^n}
\]

\[
= (-1)^l \text{Res}_{z=0} z^{-n-l-1} \sum_{k=0}^{\infty} B_k^{(n)}(n-a) \frac{z^k}{k!}
\]

\[
= (-1)^l \frac{B_{n+l}^{(n)}(n-a)}{(n+l)!},
\]

from which we obtain (3.4).

From Theorems 3.3 and 3.4 we have the following.

COROLLARY 3.5. For every integer \( l \geq 0 \) we have

\[
\zeta_n(-l,a) = (-1)^n \frac{l!}{(n+l)!} B_{n+l}^{(n)}(a).
\]

In particular, \( \zeta(-l,a) = -B_{l+1}(a)/(l+1) \).

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