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A NOTE ON OVERRINGS OF POLYNOMIAL RINGS

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Throughtout this paper, R will denote a commutative ring with identity. Let $\mathcal{F}(R)$ denote the set of all fractional ideals of R. The v-operation on R is defined as the function $F \to F_v$ of $\mathcal{F}(R)$ into $\mathcal{F}(R)$. The v-operation satisfies the properties such as (1) $(IJ)_v = (I_vJ)_v$ (2) $(\sum I_\alpha)_v = (\sum (I_\alpha)_v)_v$ (3) $\cap (I_\alpha)_v = (\cap (I_\alpha)_v)_v$ (4) $(rI)_v = r(I_v)$ for I, J, $I_\alpha \in \mathcal{F}(R)$ and a regular element r of R. When the v-operation is endlich arithmetisch brauchbar, R is called a v-ring and the ring R^v defined by the set $\{f/g | f, g \in R[X], g \text{ is regular}, (A_f)_v \subseteq (A_g)_v\}$ is called the Kronecker funtion ring of R. If the set of finite type v-ideals of R forms a group under the v-multiplication, R is called a Prüfer v-multiplication ring (abbr. PVMR). In [1, Theorem 3], Arnold and Brewer proved that the following statements are equivalent for a v-domain R : (1) R is a Prüfer v-multiplication domain (2) R^v is a quotient ring of $R[X]$ (3) Each valuation overring of R^v is of the form $(R[X])_{(P[X])}$ where R_P is a valuation overring of R (4) R^v is a flat $R[X]$-module. In proving the implication (2)$\Rightarrow$(1), they developed : [1, Lemma 1] Let R be an integral domain. If Q is a prime ideal of $R[X]$ such that $(R[X])_Q$ is a valuation ring and if $(Q \cap R)R[X] \subset Q$, then $Q \cap R = (0)$. Huckaba and Papick extended this result to additively regular rings with property A [2, Lemma 22.4] and could prove that Arnold and Brewer's result holds for this class of rings [2, Theorem 22.5]. We will give a proof of Huckaba and Papick's result which avoids the use of [2, Lemma 22.4] and which is valid even for larger class of Marot rings than the class of additively regular rings. For undefined terms and notations, the readers are referred to [2].

THEOREM (CONFER [2, Theorem 22.5]). *Let R be a Marot v-ring with property A. Then the following conditions are equivalent :*

(1) R is a PVMR.

(2) $R[X]_{(U_2)}$ is a Bezout ring.

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(3) Each regular prime ideal of $R[X]_{(\mathcal{U}_2)}$ *is extended from a prime ideal of* R.

(4) $R^v = R[X]_{(\mathcal{U}_2)}$.

(5) R^v *is a flat* $R[X]$*-module.*

(6) Each valuation overring of R^v *is of the form* $R[X]_{(P[X])}$, *where* $R_{(P)}$ *is a valuation ring.*

(7) $R[X]_{(\mathcal{U}_2)}$ *is a Prüfer ring.*

Proof. The proof for the equivalence of (1) and (4) in [2] is still valid under the new condition.

(1)⇒(3). Suppose R is a PVMR. Recently Kang [3] proved that every regular prime ideal of $R[X]_{N_v}$ is extended from R where $N_v = \{f | f \in R[X], A_f \text{ is regular, and } (A_f)_v = R\}$. In view of [2, Theorem 19.1], $(\mathcal{U}_2) = N_v$ since R has property A.

(3)⇒(2). Apply [2, Theorem 21.2].

(2)⇒(7). It is clear.

(7)⇒(5). By [4, Theorem 10.20] and [4, Exercise 11(a)(4) on page 248], R^v is a $R[X]_{(\mathcal{U}_2)}$-flat. Since $R[X]_{(\mathcal{U}_2)}$ is clearly $R[X]$-flat, R^v is $R[X]$-flat.

(5)⇒(4). For each regular maximal ideal M of $R[X]_{(\mathcal{U}_2)}$, $MR^v \subseteq R^v$. By [4, Exercise 11 on page 248], $R^v \subseteq [R[X]_{(\mathcal{U}_2)}]_{(M)}$. So $R^v \subseteq \cap_M (R[X]_{(\mathcal{U}_2)})_{(M)} = R[X]_{(\mathcal{U}_2)}$ [2, Theorem 6.1]. Hence $R^v = R[X]_{(\mathcal{U}_2)}$.

(6)⇒(5). See the proof in [2, Theorem 22.5].

(1)⇒(6). Suppose that R is a PVMR. By the equivalence of (1), (2) and (4), R^v is a Prüfer ring. By [4, Exercise 12 on page 248], each proper valuation overring of R^v is of the form $R^v_{(Q)}$, where Q is a regular prime ideal of R^v. By the equivalence of (1), (3) and (4), $R^v = R[X]_{(\mathcal{U}_2)}$ and Q is extended from R, say $Q = P[X]$ for a regular prime ideal P of R. Hence $R^v_{(Q)} = R^v_{(P[X])}$ and it is easy to see that $R_{(P)} = R[X]_{(Q)} \cap T(R)$. Since both R and $R[X]$ are Marot rings and $R[X]_{(Q)}$ is a valuation ring, the equivalence of (1) and (4) in [2, Theorem 7.7] forces $R[X]_{(Q)} \cap T(R)$ to be a valuation ring. Therefore $R_{(P)}$ is a valuation ring. □

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