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ON p—ADIC DIFFERENTIABLE FUNCTIONS

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I. INTRODUCTION

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and 0, and let $\mathbb{J}$ denote the set of $\mathbb{N}$ with the p-adic valuation $| \ |$, which is so normalized that $|p| = p^{-1}$. Hence $\mathbb{J}$ is regarded as a dense subset of the p-adic integer ring $\mathbb{Z}_p$. If a function f is continuous on $\mathbb{J}$, then f can be uniquely extended to a continuous function on $\mathbb{Z}_p$, and Mahler's expansion

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{n} \Delta^n f(0)$$

holds on $\mathbb{Z}_p$, where $|\Delta^n f(0)| \to 0$ as $n \to \infty$ and $\binom{x}{n}$ denotes the binomial coefficient in $\mathbb{Q}_p$. Moreover,

$$\Delta^n f(0) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j)$$

means the nth iterative of a linear difference operator Δ to $f(0)$, which is defined by $\Delta f(x) = f(x+1) - f(x)$. If a function f is differentiable at a point $x \in \mathbb{Z}_p$ if and only if

$$\lim_{n\to\infty} \frac{\Delta^n f(x)}{n} = 0.$$

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And C.S.Weiseman proved the following theorem: Any function f is uniformly differentiable on $\mathbb{Z}_p$ if and only if

$$n|\Delta^n f(0)| \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, it is necessary and sufficient condition for a function f to be analytic on $\mathbb{Z}_p$ that

$$|\frac{\Delta^n f(0)}{n!}| \to 0 \quad \text{as} \quad n \to \infty$$

holds in $\mathbb{C}_p$. Recently, several problems which include Mahler's expansion as a special case were treated in [2],[3],[5]. In this paper, we treat a few relationships between the magnitudes of p–differences of any differentiable function f and the p–adic values of its Mahler's coefficients $\Delta^n f(0)$. In particular, we will prove that a function f on $\mathbb{Z}_p$ is $C^{(m)}$–function if and only if $n^m|\Delta^n f(0)| \to 0$ as $n \to \infty$.

II. Results.

Let

$$f(x) = \sum_{n=0}^{\infty} \Delta^n f(0) \binom{x}{n}$$

be a continuous function on $\mathbb{Z}_p$. Now we can prove easily that

$$\Delta^n f(x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^{n+k} f(0).$$

Since

$$\begin{aligned} f(x+y) &= \sum_{m=0}^{\infty} \Delta^m f(0) \binom{x+y}{m} = \sum_{m=0}^{\infty} \Delta^m f(0) \sum_{n=0}^{m} \binom{x}{n}\binom{y}{m-n} \\ &= \sum_{n=0}^{\infty} \binom{x}{n} \sum_{k=0}^{\infty} \Delta^{n+k} f(0) \binom{y}{k} = \sum_{n=0}^{\infty} \binom{x}{n} \Delta^n f(y), \end{aligned}$$

$$\frac{f(x+y)-f(y)}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \binom{x}{n} \Delta^n f(y) = \sum_{n=1}^{\infty} \binom{x-1}{n-1} \frac{\Delta^n f(y)}{n}.$$

If $\frac{\Delta^n f(y)}{n}$ is any p-adic null sequence, then the limit

$$f^{(1)}(y) = \lim_{|x|\to 0} \sum_{n=1}^{\infty} \binom{x-1}{n-1} \frac{\Delta^n f(y)}{n}$$

is equal to

$$f^{(1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Delta^n f(y)}{n},$$

where $f^{(1)}$ denotes 1th derivative.

Thus the derivative $f^{(1)}$ exists and is continuous on $\mathbb{Z}_p$. Then all series

$$(\Delta^n f)^{(1)}(0) = \sum_{k_1=1}^{\infty} (-1)^{k_1-1} \frac{\Delta^{k_1+n} f(0)}{k_1}$$

converges for $n = 1, 2, \cdots$ and

$$f^{(1)}(x) = \sum_{n=0}^{\infty} (\Delta^n f)^{(1)}(0) \binom{x}{n}$$

for all $x \in \mathbb{Z}_p$.

Suppose that $f^{(2)}$ exists and is continuous on $\mathbb{Z}_p$. Then

$$f^{(2)}(x) = \sum_{n=0}^{\infty} (\Delta^n f)^{(2)}(0) \binom{x}{n}$$

for all $x \in \mathbb{Z}_p$, where

$$\begin{aligned}(\Delta^n f)^{(2)}(0) &= \sum_{k_1=1}^{\infty} (-1)^{k_1-1} \frac{(\Delta^{k_1+n} f)^{(1)}(0)}{k_1} \\ &= \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{k_2=1}^{\infty} \frac{(-1)^{k_2-1}}{k_2} \Delta^{n+k_1+k_2} f(0).\end{aligned}$$

Since

$$\begin{aligned} f^{(1)}(x+y) &= \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{n=0}^{\infty} \binom{x+y}{n} \Delta^{n+k_1} f(0) \\ &= \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{n=0}^{\infty} \Delta^{n+k_1} f(0) \sum_{j=0}^{n} \binom{x}{j}\binom{y}{n-j} \\ &= \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{j=0}^{\infty} \binom{x}{j} \sum_{i=0}^{\infty} \Delta^{j+k_1+i} f(0) \binom{y}{i} \\ &= \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{j=0}^{\infty} \binom{x}{j} \Delta^{j+k_1} f(y), \end{aligned}$$

$$\frac{f^{(1)}(x+y) - f^{(1)}(y)}{x} = \frac{1}{x} \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{k_2=1}^{\infty} \binom{x}{k_2} \Delta^{k_2+k_1} f(y).$$

Therefore

$$f^{(2)}(y) = \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{k_2=1}^{\infty} \frac{(-1)^{k_2-1}}{k_2} \Delta^{k_2+k_1} f(y).$$

Let

$$(\Delta^n f)^{(2)}(0) = \sum_{k_1=1}^{\infty} \frac{(-1)^{k_1-1}}{k_1} \sum_{k_2=1}^{\infty} \frac{(-1)^{k_2-1}}{k_2} \Delta^{n+k_1+k_2} f(0).$$

Then

$$f^{(2)}(y) = \sum_{n=0}^{\infty} \binom{y}{n} (\Delta^n f)^{(2)}(0).$$

Continuing this process.

If $f^{(m)}$ exists and is continuous on $\mathbb{Z}_p$, then

$$f^{(m)}(x) = \sum_{n=0}^{\infty} \binom{x}{n} (\Delta^n f)^{(m)}(0),$$

where

$$(\Delta^n f)^{(m)}(0)$$
$$= \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} \frac{(-1)^{k_1+k_2+\cdots+k_m-m}\Delta^{k_1+k_2+\cdots+k_m+n}f(0)}{k_1k_2\cdots k_m}.$$

We denote $C^{(m)}(\mathbb{Z}_p, \mathbb{C}_p) = \{f | f^{(m)} \in C(\mathbb{Z}_p, \mathbb{C}_p)\}$.

THEOREM 1. *Let*

$$f(x) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(0) \in C(\mathbb{Z}_p, \mathbb{C}_p)$$

for all $x \in \mathbb{Z}_p$. If $f \in C^{(m)}(\mathbb{Z}_p, \mathbb{C}_p)$ for $m \geq 0$, then all series

$$(\Delta^n f)^{(m)}(0) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} \frac{(-1)^{k_1+k_2+\cdots+k_m-m}\Delta^{k_1+k_2+\cdots+k_m+n}f(0)}{k_1k_2\cdots k_m}$$

converges , that is,

$$f^{(m)}(x) = \sum_{j=0}^{\infty} (\Delta^j f)^{(m)}(0) \binom{x}{j}.$$

Let $n \in \mathbb{N}$, $n \geq p^m$. Then n has the p-adic expansion $n = a_0 + a_1 p + \cdots a_t p^t$, where $a_t \neq 0$ and $t \geq 0$. Here we use R.Ahlswede, R.Bojanic method, that is,

$$\begin{aligned} n &= k_1 + k_2 + \cdots + k_m + j \\ p^t &= k_1 + k_2 + \cdots + k_m \\ p^{t-1} &= k_2 + \cdots + k_m \\ &\cdots \\ p^{t-m+1} &= k_m \end{aligned}$$

Thus

$$p^t - p^{t-1} = k_1, p^{t-1} - p^{t-2} = k_2, \cdots, p^{t-m+2} - p^{t-m+1} = k_m.$$

Therefore

$$|k_1| = p^{-t+1}, |k_2| = p^{-t+2}, \cdots, |k_m| = p^{-t+m-1}.$$

By the above results,

$$|\frac{\Delta^{k_1+k_2+\cdots+k_m+j}f(0)}{k_1k_2\cdots k_m}| = |\Delta^n f(0)|p^{mt-\frac{m(m-1)}{2}}.$$

Then, for $p^t \leq n < p^{(t+1)}$,

$$\begin{aligned}|\frac{\Delta^{k_1+k_2+\cdots+k_m+j}f(0)}{k_1k_2\cdots k_m}| &= |\Delta^n f(0)|p^{mt+m}p^{-m-\frac{m(m-1)}{2}} \\ &= |\Delta^n f(0)|p^{m(t+1)}p^{-\frac{m(m+1)}{2}} \\ &\geq |\Delta^n f(0)|n^m p^{-\frac{m(m+1)}{2}}.\end{aligned}$$

Thus we have the following result.

THEOREM 2. *Let*

$$f(x) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(0) \in C(\mathbb{Z}_p, \mathbb{C}_p)$$

for all $x \in \mathbb{Z}_p$. *If*

$$(\Delta^n f)^{(m)}(0) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} \frac{(-1)^{k_1+k_2+\cdots k_m} \Delta^{k_1+k_2+\cdots+k_m+j} f(0)}{k_1k_2\cdots k_m}$$

converges, then

$$f^{(m)}(x) = \sum_{j=0}^{\infty} (\Delta^j f)^{(m)}(0) \binom{x}{j}$$

exists and is continuous from $\mathbb{Z}_p$ *to* $\mathbb{C}_p$. *Moreover,* $n^m|\Delta^n f(0)| \to 0$ *as* $n \to \infty$.

In particular, if $n^m|\Delta^n f(0)| \to 0$ as $n \to \infty$, then

$$|\frac{\Delta^{k_1+k_2+\cdots+k_m+j}f(0)}{k_1k_2\cdots k_m}| \to 0 \quad \text{as} \quad n \to \infty$$

because of

$$\begin{aligned}&|\frac{\Delta^{k_1+k_2+\cdots+k_m+j}f(0)}{k_1k_2\cdots k_m}|\\&\leq (k_1k_2\cdots k_m)|\Delta^{k_1+k_2+\cdots+k_m+j}f(0)|\\&\leq (k_1+k_2+\cdots+k_m+j)^m|\Delta^{k_1+k_2+\cdots+k_m+j}f(0)|.\end{aligned}$$

Therefore we obtain the following result.

THEOREM 3. *If*

$$f(x)=\sum_{j=0}^{\infty}\binom{x}{j}\Delta^j f(0)\in C(\mathbb{Z}_p,\mathbb{C}_p)$$

for all $x\in\mathbb{Z}_p$ *and* $m\geq 0$. *Then*

$$f\in C^{(m)}(\mathbb{Z}_p,\mathbb{C}_p)\quad \text{if and only if}\quad n^m|\Delta^n f(0)|\to 0\quad \text{as}\quad n\to\infty.$$

DEFINITION. ([1]) Let f be a p–adic valued function defined on $\mathbb{J}$ by

$$\max_{k\in\mathbb{J}}|f(k+p^t)-f(k)|=O(p^{-\alpha t}).$$

Then f is called α–Lipschitz function.

THEOREM 4. *Let*

$$f(x)=\sum_{j=0}^{\infty}\binom{x}{j}\Delta^j f(0)\in C(\mathbb{Z}_p,\mathbb{C}_p)$$

for all $x\in\mathbb{Z}_p$. *If* $n^\alpha|\Delta^n f(0)|\to 0$ *as* $n\to\infty$ *for* $\alpha>1$ *with* $\alpha\notin\mathbb{N}$, *then* $f^{([\alpha])}$ *is* $\alpha-[\alpha]$*–Lipschitz function, where* $[\cdot]$ *is Gauss' symbol.*

Proof. Since

$$f^{([\alpha])}(n+p^t)-f^{([\alpha])}(n)=\sum_{k=0}^{n}(\sum_{j=1}^{p^t}\binom{p^t}{j}(\Delta^{k+j}f)^{([\alpha])}(0))\binom{n}{k},$$

$$|f^{([\alpha])}(n+p^t) - f^{([\alpha])}(n)| \le \max_{k\in\mathbb{J}} |\sum_{j=1}^{p^t}(\Delta^{k+j}f)^{([\alpha])}(0)\binom{p^t}{j}|.$$

And since $n^\alpha|\Delta^n f(0)| \to 0$ as $n \to \infty$,

$$|\sum_{j=1}^{p^t}(\Delta^{k+j}f)^{([\alpha])}(0)\binom{p^t}{j}| \le p^{-t} \max_{1\le j\le p^t} |\frac{(\Delta^{k+j}f)^{([\alpha])}(0)}{j}|$$
$$\le p^{-t} \max_{1\le j\le p^t} j(k+j)^{[\alpha]}|\Delta^{k+j}f(0)|$$
$$= p^{-t} \max_{1\le j\le p^t} j(k+j)^{-\alpha+[\alpha]}(k+j)^{\alpha-[\alpha]}(k+j)^{[\alpha]}|\Delta^{k+j}f(0)|$$
$$\le Mp^{-t} \max_{1\le j\le p^t} j^{1-[\alpha]} \le Mp^{-(\alpha-[\alpha])t} \quad \text{for some constant} \quad M>0.$$

Thus we have

$$|f^{([\alpha])}(n+p^t) - f^{([\alpha])}(n)| \le Mp^{-(\alpha-[\alpha])t}.$$

for some constant $M > 0$.

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