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ON SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS. IV

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Let T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0). \tag{1.2}$$

A function $f(z) \in T$ is said to be in the class $S^*(\alpha, \beta, \mu)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\mu \frac{zf'(z)}{f(z)} + 1 - (1+\mu)\alpha} \right| < \beta \tag{1.3}$$

for some $\alpha(0 \leq \beta \leq 1)$, $\mu(0 \leq \mu \leq 1)$, and for all $z \in U$. Further $f(z) \in T$ is said to be in the class $C^*(\alpha, \beta, \mu)$ if ans only if $zf'(z) \in S^*(\alpha, \beta, \mu)$. The classes $S^*(\alpha, \beta, \mu)$ and $C^*(\alpha, \beta, \mu)$ were studied by Owa and Aouf [7] and Aouf [1].

We note that:

(i) $S^*(\alpha, \beta, 1) = S^*(\alpha, \beta)$ and $C^*(\alpha, \beta, 1) = C^*(\alpha, \beta)$ were studied by Gupta and Jain [2], Owa [6] and Kumar and Shukla [3].

(ii) $S^*(\alpha, 1, 1) = S^*(\alpha)$ and $C^*(\alpha, 1, 1) = C^*(\alpha)$ were studied by Silverman [8].

In order to show our results, we need the following lemmas given by Owa and Auf [7].

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LEMMA 1. *A function $f(z)$ defined by (1.2) is in the class $S^*(\alpha,\beta,\mu)$ if and omly if*

$$\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_n \leq (1+\mu)\beta(1-\alpha), \tag{1.4}$$

where

$$D(n,\alpha,\beta,\mu) = (n-1) + \beta[\mu n + 1 - (1+\mu)\alpha]. \tag{1.5}$$

The result is sharp.

LEMMA 2. *A function $f(z)$ defined by (1.2) is in the class $C^*(\alpha,\beta,\mu)$ if and only if*

$$\sum_{n=2}^{\infty} nD(n,\alpha,\beta,\mu)a_n \leq (1+\mu)\beta(1-\alpha). \tag{1.6}$$

The result is sharp.

2. Closure Theorems

Let the functions $f_j(z)$ be defined, for $j = 1,2,\cdots,m$, by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}z^n \qquad (a_{n,j} \geq 0) \tag{2.1}$$

for $z \in U$.

We shell prove the following results for the closure of functions the classes $S^*(\alpha,\beta,\mu)$.

THEOREM 1. *Let the functions $f_j(z)(j=1,2,\cdots,m)$ defined by (2.1) be in the class $S^*(\alpha,\beta,\mu)$. Then the function $h(z)$ defined by*

$$h(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{2.2}$$

also belongs to the class $S^*(\alpha,\beta,\mu)$, *where*

$$b_n = \frac{1}{n}\sum_{j=1}^{m} a_{n,j}. \tag{2.3}$$

Proof. Since $f_j(z) \in S^*(\alpha,\beta,\mu)$, it follows from Lemma 1 that

$$\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_{n,j} \le (1+\mu)\beta(1-\alpha),\ j = 1,2,\cdots,m.$$

Therefore

$$\begin{aligned}\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)b_n &= \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)\{\frac{1}{M}\sum_{j=1}^{m} a_{n,j}\}\\ &\le (1+\mu)\beta(1-\alpha).\end{aligned} \tag{2.4}$$

Hence by Lemma 1, $h(z) \in S^*(\alpha,\beta,\mu)$. Thus we have the theorem.

By using Lemma 2, we have

THEOREM 2. *Let the function* $f_j(z)$ $(j = 1,2,\cdots,m)$ *defined by (2.1) be in the class* $C^*(\alpha,\beta,\mu)$. *Then the function* $h(z)$ *defined by (2.2) also belongs the class* $C^*(\alpha,\beta,\mu)$ *under the condition (2.3).*

THEOREM 3. *Let the functions* $f_j(z)(j = 1,2,\cdots,m)$ *defined by (2.1) be in the class* $S^*(\alpha,\beta,\mu)$. *Then the function* $h(z)$ *defined by*

$$h(z) = \sum_{j=1}^{m} d_j f_j(z) \qquad (d_j \ge 0) \tag{2.5}$$

is also in the same class $S^*(\alpha,\beta,\mu)$, *where*

$$\sum_{j=1}^{m} d_j = 1. \tag{2.6}$$

Proof. According to the definition of $h(z)$, we can write that

$$h(z) = z - \sum_{n=2}^{\infty}[\sum_{j=1}^{m} d_j a_{n,j}]z^n. \tag{2.7}$$

By means of Lemma 1, we have

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_{n,j} \leq (1 + \mu)\beta(1 - \alpha) \tag{2.8}$$

for every $j = 1, 2, \cdots, m$. Hence we can observe that

$$\begin{aligned} \sum_{n=2}^{\infty} D(\alpha, \beta, \mu)[\sum_{j=1}^{m} d_j a_{n,j}] &= \sum_{j=1}^{m} d_j[\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_{n,j}] \\ &\leq [\sum_{j=1}^{m} d_j](1 + \mu)\beta(1 - \alpha) = (1 + \mu)\beta(1 - \alpha) \end{aligned} \tag{2.9}$$

which implies that $h(z) \in S^*(\alpha, \beta, \mu)$. Thus we have the theorem.

By using Lemma 2, we have

THEOREM 4. *Let the functions $f(z)$ defined by (2.1) be in the class $C^*(\alpha, \beta, \mu)$ for every $j = 1, 2, \cdots, m$. Then the function $h(z)$ defined by (2.5) is also belongs to the same calss $C^*(\alpha, \beta, \mu)$ under the condition (2.6).*

THEOREM 5. *Let the function $f_1(z)$ defined by (2.1) be in the class $S^*(\alpha, \beta, \mu)$ and the function $f_2(z)$ defined by (2.1) be in the calss $C^*(\alpha, \beta, \mu)$. Then the function $k(z)$ defined by*

$$k(z) = z - \frac{2}{3}\sum_{n=2}^{\infty}(a_{n,1} + a_{n,2})z^n \tag{2.10}$$

is in the calss $S^*(\alpha, \beta, \mu)$.

Proof. Since $f_1(z) \in S^*(\alpha, \beta, \mu)$ and $f_2(z) \in C^*(\alpha, \beta, \mu)$, by using Lemma 1 and 2 we get, respectively,

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, mu) a_{n,1} \leq (1+\mu)\beta(1-\alpha) \tag{2.11}$$

and

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_{n,2} \leq \frac{(1+\mu)\beta(1-\alpha)}{2}. \tag{2.12}$$

Therefore, we have

$$\frac{2}{3} \sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu)(a_{n,1} + a_{n,2}) \leq (1+\mu)\beta(1-\alpha) \tag{2.13}$$

which implies that $k(z) \in S^*(\alpha, \beta, \mu)$, and the proof of Theorem 5 is thus completed.

3. Integral Operators

THEOREM 6. *Let the function* $f(z)$ *definede by (1.2) be in the class* $S^*(\alpha, \beta, \mu)$ *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \tag{3.1}$$

also belongs to the class $S^*(\alpha, \beta, \mu)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \tag{3.2}$$

where

$$b_n = [\frac{c+1}{c+n}]a_n. \tag{3.3}$$

Therefore,

$$\begin{aligned}\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)b_n &= \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)[\frac{c+1}{c+n}]a_n \\ &\leq \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_n \leq (1+\mu)\beta(1-\alpha),\end{aligned} \tag{3.4}$$

since $f(z) \in S^*(\alpha,\beta,\mu)$. Hence, by Lemma 1, $F(z) \in S^*(\alpha,\beta,\mu)$.

THEOREM 7. *Let c be a real number such that $c > -1$. If $F(z) \in S^*(\alpha,\beta,\mu)$, then the function $f(z)$ defined by (3.1) is univalent in $|z| < R^*$, where*

$$R^* = \inf_n [\frac{D(n,\alpha,\beta,\mu)(c+1)}{n(1+\mu)\beta(1-\alpha)(c+n)}]^{\frac{1}{n-1}} \qquad (n \geq 2). \tag{3.5}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0)$. It follows from (3.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}[z^c F(z)]'}{(c+1)} (c > -1) \\ &= z - \sum_{n=2}^{\infty} [\frac{c+n}{c+1}]a_n z^n. \end{aligned} \tag{3.6}$$

In order to obtain the required result it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$.

Now $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1} < 1. \tag{3.7}$$

According to Lemma 1, we have

$$\sum_{n=2}^{\infty}\frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}a_n \leq 1. \tag{3.8}$$

Hence (3.7) will be true if

$$\frac{n(c+n)|z|^{n-1}}{(c+1)} < \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}$$

or if

$$|z| < [\frac{D(n,\alpha,\beta,\mu)(c+1)}{n(1+\mu)\beta(1-\alpha)(c+n)}]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.9}$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows it we take

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)(c+n)}{D(n,\alpha,\beta,\mu)(c+1)}z^n \quad (n \geq 2) \tag{3.10}$$

THEOREM 8. *Let c be areal number such that $c > -1$. If $F(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq))$ belongs to the class $S^*(\alpha,\beta,\mu)$, than the function $f(z)$ defined by (3.1) is starlike of order $\sigma (0 \leq \sigma < 1)$ in $|z| < r^*(\sigma,\alpha,\beta,\mu)$, where*

(3.11)

$$r^*(\sigma,\alpha,\beta,\mu) = \inf_n [[\frac{1-\sigma}{n-\sigma}][\frac{c+1}{c+n}]\frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}]^{\frac{1}{n-1}} (n \geq 2).$$

The result is sharp.

Proof. In order to establish the required result it suffices to show that

$$|\frac{zf'(z)-1}{f(z)}| < (1-\sigma) \ \ in \ |z| < r^*(\sigma,\alpha,\beta,\mu).$$

Now

$$\begin{aligned} \frac{|zf'(z)}{f(z)} - 1| &= |\frac{-\sum_{n=2}^{\infty}(n-1)[\frac{c+n}{c+1}]a_n z^{n-1}}{1-\sum_{n=2}^{\infty}[\frac{c+n}{c+1}]a_n z^{n-1}}| \\ &\leq \frac{\sum_{n=2}^{\infty}(n-1)[\frac{c+n}{c+1}a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}[\frac{c+n}{c+1}]a_n|z|^{n-1}} \\ &< (1-\sigma), \end{aligned}$$

provided

$$\sum_{n=2}^{\infty}[\frac{n-\sigma}{1-\sigma}][\frac{c+n}{c+1}]a_n|z|^{n-1} < 1. \tag{3.13}$$

By using (3.9), the ineqality (3.13) holds if

$$[\frac{n-\sigma}{1-\sigma}][\frac{c+n}{c+1}]|z|^{n-1} < \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)} \quad (n \geq 2)$$

or if

$$|z| < \{[\frac{1-\sigma}{n-\sigma}][\frac{c+1}{c+n}]\frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}\}^{\frac{1}{n-1}} (n \geq 2).$$

Hence, $f(z) \in S^*$ in $|z| < r^*(\sigma,\alpha,\beta,\mu)$. Sharpness follows if we take the function $F(z)$ given by

$$F(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(n,\alpha,\beta,\mu)} z^n (n \geq 2). \tag{3.14}$$

REMARK. *Putting $c = \mu = 1$ in Theorem 8, we get the result of Kumar and Shukla [3, Theorem 2].*

THEOREM 9. *Let the function $f(z)$ be defined by (1.2). If $f(z) \in S^*(\alpha,\beta,\mu)$, then the function $F(z)$ defined by (3.1) belongs to $S^*(\sigma)$, where*

$$\sigma = \frac{(c+2)+\beta[(2\mu-c)+c(1+\mu)\alpha]}{(c+2)+\beta[(3+c)\mu+1-(1+\mu)\alpha]}. \tag{3.15}$$

The result is sharp. Further, the converse need not be true.

Proof. Let $F(z) = z - \sum_{n=2}^{\infty} b_n z^n \in S^*(\alpha)$, where b_n is given by (3.3), then, by Lemma 1, it holds if and only if

$$\sum_{n=2}^{\infty}[\frac{n-\sigma}{1-\sigma}]b_n \leq 1. \tag{3.16}$$

Thus we have to find the largest value of σ so that the inequality (3.16) holds. Now by using (3.8), (3.16) holds if

$$[\frac{n-\sigma}{1-\sigma}]b_n \leq \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}a_n(n \geq 2)$$

or if

$$[\frac{n-\sigma}{1-\sigma}][\frac{c+1}{c+n}] \leq \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}(n \geq 2), \tag{3.17}$$

which is eqalivalent to

$$\begin{aligned} \sigma &\leq \frac{(c+n)D(n,\alpha,\beta,\mu)-(c+1)n(1+\mu)\beta(1-\alpha)}{(c+n)D(n,\alpha,\beta,\mu)-(c+1)(1+\mu)\beta(1-\alpha)} \\ &= \sigma_n, \quad say\ ,(n \geq 2). \end{aligned} \tag{3.18}$$

It is easy to verify that σ_n is increasing function of $n(n \geq 2)$. Therefore $\sigma = inf_{n\geq 2}\sigma_n = \sigma_2$ and, hence

$$\sigma = \frac{(c+2)+\beta[(2\mu-c)+c(1+\mu)\alpha]}{(c+2)+\beta[(3+c)\mu+1-(1\mu)\alpha]}.$$

To show the sharpness we take the function $f(z)$ given by

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(2,\alpha,\beta,\mu)}z^2. \tag{3.19}$$

Then

$$F(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{(c+2)D(2,\alpha,\beta,\mu)}z^2, \tag{3.20}$$

and, therefore

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{(c+2)D(2,\alpha,\beta,\mu)-2(c+1)(1+\mu)\beta(1-\alpha)z}{(c+2)D(2,\alpha,\beta,\mu)-(c+1)(1+\mu)\beta(1-\alpha)z} \\ &= \frac{(c+2)+\beta[(2\mu-c)+c(1+\mu)\alpha]}{(c+2)+\beta[(3+c)\mu+1-(1+\mu)\alpha]}, \text{ for } z=1. \end{aligned}$$

Hence, the result is sharp.

We now show that the converse of the theorem need not be true. To this end we consider the function

$$F(z) = z - [\frac{1-\sigma}{3-\sigma}]z^3. \tag{3.21}$$

Lemma 1 guarantees that $F(z) \in S^*(\sigma)$. But the corresponding function

$$f(z) = z - \frac{(c+3)(1-\sigma)}{(c+1)(3+\sigma)}z^3 \tag{3.22}$$

does not belong to $S^*(\alpha, \beta, \mu)$, since, for this $f(z)$ the coefficient inequality of Lemma 1 is not satisfied.

COROLLARY 1. *Let the function $f(z)$ be defined by (1.2). If $f(z) \in S^*(\alpha)(0 \leq \alpha < 1)$, then the function $F(z)$ defined by (3.1) belongs to the class $S^*(\frac{2+c\alpha}{c+3-\alpha})$. The result is sharp. The converse need not be true.*

REMARKS.

(1) *Putting $c = \mu = 1$ in Theorem 9, we get the result of Kumar and Shukla [3,Theorem 1].*

(2) *$\alpha = 0$ and $c+1$ in Corollary 1, we get the result of Kumar and Shukla [3,Corollary 1].*

4. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [9].

DEFINITION 1. *For real numbers $\rho > 0$, δ and η, the fractional integral operator $I_{0,z}^{\rho,\delta,\eta}$ is defined by*

$$(4.1)\quad I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{z^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^z (z-t)^{\sigma-1} F(\sigma+\delta, -\eta; \sigma; 1-\frac{t}{z}) f(t) dt$$

where $f(z)$ is an analytic function in a simple connected region of the z-plain containing yhe origin with the order

$$f(z) = O(|z|^{\epsilon}), z \longrightarrow 0,$$

where

$$\epsilon > Max(0, \delta\eta) - 1,$$

$$(4.2)\qquad F(a, b : c : z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where $(\mu)_n$ is pochhammer symbol defined by
(4.3)

$$(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \begin{cases} 1 & (n=0) \\ \mu(\mu+1)\cdots(\mu+n-1) & (n \in N = (\{1, 2, \cdots\}), \end{cases}$$

and the multiplicity of $(z-t)^{\rho-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

REMARK. *For $\delta = -\rho$, we note that*

$$I_{0,z}^{\rho,-\rho,\eta} f(z) = D_z^{-\rho} f(z),$$

where $D_z^{-\rho} f(z)$ is the fractional interal of order ρ of $f(z)$ which was *introduce by* Owa *([4],[5]).*

In order to prove our results for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [9].

LEMMA 3. *If $\rho > 0$ and $n > \delta - \eta - 1$, then*

$$I_{0,z}^{\rho,\delta,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\rho+\eta+1)} z^{n-\delta}. \tag{4.4}$$

With the aid of Lemma 3, we have

THEOREM 10. *Let $\rho > 0$, $\delta < 2$, $\rho + \eta > -2$, $\delta - \eta < 2$, and $\delta(\rho+\eta) \leq 3\rho$. If $f(z) \in T$ is in the class $S^*(\alpha, \beta, \mu)$, then*

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \{1 - \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\delta,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\} \tag{4.5}$$

and

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \{1 + \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\delta,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\} \tag{4.6}$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U(\delta \leq 1) \\ U - \{0\}(\delta > 1) \end{cases}$$

The eqalities in (4.5) and (4.6) are attained by the function

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(2,\alpha,\beta,\mu)} z^2. \tag{4.7}$$

Proof. By using Lemma 3, we have

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\rho+\eta+1)} a_n z^{n-\delta}. \tag{4.8}$$

Letting

$$H(z) = \frac{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}{\Gamma(2-\delta+\eta)} Z^{\delta} I_{0,z}^{\rho,\delta,\eta} f(z)$$

$$z - \sum_{n=2}^{\infty} h(n) a_n z^n, \tag{4.9}$$

where

$$h(z) = \frac{(2-\delta+\eta)_{n-1}(1)_n}{(2-\delta)_{n-1}(2+\rho+\eta)_{n-1}} (n \geq 2) \tag{4.10 ,}$$

we can see that $h(n)$ is non-increasing for intergers $n \geq 2$, and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\rho+\eta)}. \tag{4.11}$$

Since $f(z) \in S^*(\alpha, \beta, \mu)$, Lemma 1 implies that

$$D(2, \alpha, \beta, \mu) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_n \leq (1+\mu)\beta(1-\alpha),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{D(2, \alpha, \beta, \mu)}. \tag{4.12}$$

Therefore, by using (4.11) and (4.12), we have

$$|H(z)| \geq |z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq |z| - \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2, \delta, \beta, \mu)(2-\delta)(2+\rho+\eta)} |z|^2 \tag{4.13}$$

and

$$|H(z)| \geq |z| + h(2)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq |z| + \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2, \delta, \beta, \mu)(2-\delta)(2+\rho+\eta)} |z|^2 \tag{4.14}$$

This completes the proof of Theorem 10.

THEOREM 11. *Let $\rho > 0, \delta < 2, \rho+\eta > -2, \delta-\eta < 2$, and $\delta(\rho+\eta) \leq 3\rho$. If $f(z) \in T$ is in the class $C^*(\alpha,\beta,\mu)$, then*

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \{1 - \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\delta,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\} \tag{4.15}$$

and

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \{1 + \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\delta,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\} \tag{4.16}$$

for $z \in U_0$, where U_0 is defined Theorem 10. The equalities in (4.15) and (4.16) are attained by the function

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{2D(2,\alpha,\beta,\mu)} z^2. \tag{4.17}$$

REMARKS.

(1) *Taking $\rho = -\delta = k$ in Theorem 10 and 11, we get the results of Theorem 3 and 4 obtained by Aouf [1], respectively.*

(2) *Putting $\mu = 1$ in Theorem 10 and 11, we get the corresponding results for the classes $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ studied by Gupta and Jain [2].*

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