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NÉRON FUNCTIONS ON PROJECTIVE CURVES

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1. Introduction

Weil in 1928 proved Poincaré's conjecture in higher dimension, and also dealt with abelian varieties over abitrary number fields. The extension to number fields and abelian varieties affered serious problems, because at that time among other things, it was not clear how to estimate the size of a point. Thus Siegl defined technically by a notion called height to define the size in a fairly general context, involving several variables and more general domains than the integer or rational numbers. The concept of the height is the basic instrument in number - theoretical procedures of "counting" and "des ent" of solutions. The hight is utilizedin the proofs of theorems of Mordell-Weil and Siegel. On the other hand, Néron, in his original construction of the canonical height, proceed by constructing a local height pairing, or Néron pairing for each absolute value $\nu \in M_K$ and then he formed the global canonical height by taking the sum of thw local heights. The theory of local height function is important in the study of the the more delicate properties of the canonical height, and gives the natural tool to express results in Diophantine approximations. These functions are intersection multiplicities at the finite place and essentially Green's functions in one form or another at the infinite ν.

In this thesis we will show that the Weil functions on any projective curve C associated with a divisor of degree zero whose supports are disjoint from the singular points are also normalized, that is they are uniquely determined up to constant functions. Again such functions are called Néron functions. Having normalized the Néron functions up to constants, we can get rid of these constants if we evaluate these functions by additivity on zero-cycles of degree zero on C. We then obtain a bilinear pairing between zero cycles of degree zero on C whose

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supports are disjoint from the singular points and zero cycles of degree zero on C. In this thesis, K is a field with a proper set of absolute values $_K$. We define

$$\nu(a) = -log|a|_\nu \text{ for } a \in K.$$

If $x = (x_1, \ldots, x_n)$ is a finite family of elements of K, we denote

$$|x|_\nu = max_i |x_i|_\nu, \quad \nu(x) = -log|x|_\nu = inf_i \nu(x_i).$$

We agree $\nu(0) = \infty$.

2. Weil functions on a projective varieties.

Let D be a Cartier divisor on V. By a Weil function associated with D we mean a function

$$\lambda_D : V(F^a) - supp(D) \to R$$

having the following property. If D is represented by a pair (U,f), then there exists a locally bounded continuous function

$$\alpha : U \times M \to R$$

such that for $P \in U - supp(D)$, we have

$$\lambda_D(P, \nu) = \nu \circ f(P) + \alpha(P, \nu).$$

The continuity of α is ν-continuity, not Zariski topology continuity. A rational function f on V defines a Cartier divisor which is called principal.Therefore the function

$$\lambda_f : V - \mathrm{supp}(f) \to R$$

defined by

$$\lambda_f(P, \nu) = \nu \circ f(P)$$

is a Weil function whose Cartier divisor is represented by f everywhere.

On a projective variety V, for each Cartier divisor D on V, there exists a Weil function λ associated with D. The association $D \mapsto \lambda_D$ is a homomorphism of

$$\text{Cartier divisors on } V \to \text{Weil functions on } V/O(1).$$

Weil functions behave functorially in the following sense. Let $f : X' \to X$ be a morphism defined over F, and let D be a Cartier divisor on X. Assume$f(X')$ is not contained in the support of D, then $\lambda_D \circ f$ is a Weil function on X' associated with f^*D. Therefore we have

$$\lambda_{f^*D} = \lambda_D \circ f + O(1).$$

Furthermore, if D is a positive Cartier divisor, then the Weil function λ_D can be extended to all of V by defining $\lambda_D(x) = \infty$ at points in the support of D. Therefore we have : if λ is a weil function on a projective variety whose Cartier divisor is positive, then there exists an M_K constant function γ such that

$$\lambda(P, \nu) \geq \gamma(\nu)$$

for all points of $V \times M$.

EXAMPLE 2.1. *Let $V = P^1$ and let $D = (\infty)$ be the point at infinity. Then*

$$\lambda(z) = max(0, log|z|)$$

is the Weil function associated with D.

DEFINITION. *Let V be a variety defined over a field F with a proper set of absolute values satisfying the product formula, and let λ be a Weil function on V and let (λ) be the Cartier divisor associated with the Weil function λ. For $P \in V(F^a) - supp((\lambda))$, we define the associated height*

$$h_\lambda(p) = \frac{1}{[K : F]} \sum_{\nu \in M_k} n_\nu \lambda(P, \nu)$$

for any finite extension K of F over which P is rarional.

REMARK2.2. *(1) If two Weil functions λ, λ' are linearly equivalent, that is there exists $f \in K(V)$ and a constant function γ such that $\lambda = \lambda' + \lambda_f + \gamma$ and we denote by $\lambda \sim \lambda'$, then by the product formula we have*

$$h_\lambda(P) = h'_\lambda(P) + constant.$$

(2) Given a Cartier divisor X and a point $P \in V$, there exists$f \in K(V)$ such that $P \notin supp(X - (f))$. Suppose $X = (\lambda)$ be the Cartier divisor associated with the Weil function λ. Put $\lambda' = \lambda - \lambda_f$ and define $h_\lambda(p) = h_{\lambda'}(P)$. This shows that h_λ is defined up to constant for all $P \in V(F^a)$.

LEMMA 2.3. *Let λ, λ' be two Weil functions associated with the same Cartier divisor on a projective variety V. Then $h_\lambda - h_{\lambda'}$ is bounded.*

proof. For all $(P, \nu) \in V(F^a) \times M(K^a)$

$$|\lambda(P,\nu) - \lambda'(P,\nu)| \leq \gamma(\nu)$$

for some positive M_K-constant function γ. Thus

$$|h_\lambda - h_{\lambda'}| \leq \frac{1}{[K:F]} \sum_{\nu \in M_K} n_\nu \gamma(\nu).$$

LEMMA 2.4. *Let λ_i $(i = 1, \ldots, m)$ be Weil functions on a projective variety V whose Cartier divisors have the form $(\lambda_i) = Y + X_i$ with $X_i \geq 0$ for all i and such that the supports of $X_1, \ldots, X_m$ have no points in common. Then the functios defined by*

$$\lambda(P,\nu) = \inf_i \lambda_i(P,\nu)$$

for $P \notin supp(Y)$ is a Weil function associated with the Cartier divisor Y.

proof. See [6].

THEOREM 2.5. *Let D be any Cartier divisor on a projective variety V and let λ be a Weil function associated with me Cartier divisor on a projective variety V. Then $h_\lambda - h_{\lambda'}$ is bounded.*

Proof. Let $(f_0, f_1, \cdots, f_m)$ be a generators of $L(D)$ and let

$$\psi = (f_0, f_1, \cdots f_m) : V \to P^m$$

be the corresponding morphism of V into P^m. then there exists a positive divisors X_i such that $(f_i) = X_i - D$. Now

$$h_\psi(P) = h(f_0(P), f_1(P), \cdots, f_m(P))$$
$$= \frac{1}{[K:F]} \sum_{\nu \in M_k} n_\nu suplog|f_i(P)|_\nu$$

But $suplog|f_i(P)|_\nu = -inf \nu \circ f_i(P)$ be the Weil function associated with the Cartier divisor D, Thus $h_\psi - h_\lambda$ is defined.

THEOREM 2.6. *Let λ be a Weil function whose Cartier divisor is positive on a projective variety V. Then there exists a constant K such that*

$$h_\lambda(P) \geq K \quad for\ all P \in V.$$

Proof. There exists a constant γ such that

$$\lambda(P, \nu) \geq \gamma(\nu) for\ all (P, \nu) \in V \times M.$$

Thus

$$h_\lambda(P) = \frac{1}{[K:F]} \sum_{\nu \in M_K} n_\nu \lambda(P, \nu)$$
$$\geq \frac{1}{[K:F]} \sum_{\nu \in M_K} n_\nu(\gamma(\nu)) \equiv K.$$

3. Néron functions on Projective Curves.

We have shown that on a projective variety, Weil functions associated with the same divisor are defined only up to a bounded function. Observe that if φ is a rational function on a variety V, then φ determines a Weil function

$$\lambda_{\varphi}(P) = -log|\varphi(P)|_{\nu}.$$

If V is complete, then the divisor of a rational function determines the function up to nonzero multiplicative constants, so the Weil function defined above is determined up to an additive constant. The normalization of Weil functions on an abelian variety is as follows:

Let A be an abelian variety defined over K, and let Γ be the group of M_K-constant functions. Then to each divisor D on A, there exists a Weil function λ_D associated with D, and uniquely determined mod Γ by the following properties (see [6] and [8]).

(1) The association $D \mapsto \lambda_D$ is a homomorphism mod Γ.

(2) If $D = (\varphi)$ is principal, then $\lambda_D = \lambda_{\varphi} + mod\Gamma$.

(3) Let $a \in A(F^a)$, and let T_a be the translation by a, and put $D_a = T_a(D)$. Then there exists a constant $\gamma_{a,D}$ such that

$$\lambda_{D_a} = \lambda_D \circ T_{-a} + \gamma_{a,D}.$$

(4) Let $f : B \to A$ be a homomorphism of abelian varieties over F. then we have

$$\lambda_{f^*D} = \lambda_D \circ f mod\Gamma.$$

Functions normalized as such are called Néron functions. the Weil functions on arbitrary complete non-singular varieties associated with divisors are obtained by pull back from abelian varieties. However, the divisors are restricted those which are algebraically equivalent to zero. Again such Weil functions are called Néron functions. Having normalized the Néron functions up to additive constants, we can get rid of these constants if we evaluate these functions by additivity on 0-cycles of degree zero on an abelian variety or non-singular projective variety. We then obtain a bilinear pairing between divisors (lgebraically equivalent to zero on an arbitrary variety) and 0-cycles of degree zero. This pairing is called the Néron Pairing or Néron symbol.

For singular projective curve, we have:

LEMMA 3.1. *Let $f: C' \to C$ be the normalization of C, and D be a Cartier divisor on C. Then*

$$deg(D) = deg(f^* D)$$

proof. See [2].

THEOREM 3.2. *Let C be a projective curve defined over a field K and S be the set of singular points on C. Let $Z_0{}'(C)_K$ be the set of zero cycles of degree zero on C rational over K whose supports are disjoint from S. Then for each $D \in Z_0{}'(C)_K$, λ_D is uniquely determined up to constant functions Γ satisfying*

1) *The association $D \mapsto \lambda_D$ is a homomorphism* mod Γ
2) *If $D = (f)$ is principal, then $\lambda_D = \lambda_f \mod \Gamma$*
3) *If $\psi : V \to W$ be a morphism of projective curves defined over K, and let $Y \in Z_0'(W)$ such that $\psi^{-1} \in Z_0'(V)$, then*

$$\lambda_{\psi^{-1}(Y)} = \lambda_Y \circ \psi \mod \Gamma$$

proof. Let $f : C' \to C$ be the normalization, and let λ, λ' be two Weil functions corresponding to D. Then $\lambda \circ f$ and $\lambda' \circ f$ are Weil functions on C' associated with the Cartier divisor $f^* D \in Z_0(C')$. Since C' is nonsingular and $f^* D$ is algebraically equivalent to zero, we have

$$\lambda \circ f = \lambda' \circ f \mod \Gamma$$

For $Q \in C$, let $P_1, \cdots, P_t$ be distinct points in $f^{-1}(Q)$, then

$$\lambda(Q) = \lambda \circ f(P_1) = \cdots = \lambda \circ f(P_t)$$

and

$$\lambda' \circ f(P_1) = \cdots = \lambda' \circ f(P_t)$$

Therefore, $\lambda = \lambda'$ mod Γ.

COROLLARY 3.3. *Let C be a projective curve defined over K such that $C(K)$ is zariski dense in C. Let $\alpha \in Z_0'(C)$, $\beta \in Z_0(C)$ be two zero cycles with disjoint supports. Then there exists unique pairing $< \alpha, \beta >_\nu$ satisfying the following properties.*

(1) *The pairing is bilinear.*

(2) *If $\alpha = (f)$ is principal, then $< \alpha, \beta >_\nu = \nu \circ f(\beta)$.*

(3) *If $\beta \in Z_0'(C)$, then $< \alpha, \beta >_\nu = < \beta, \alpha >_\nu$.*

(4) *The function $x \mapsto < \alpha, (x) - (x_0) >_\nu$ from $C(K) - supp(\alpha)$ to R is continuous and locally bounded.*

4. Néron Functions on Riemann Surfaces.

In this chapter we give some explicit formulas for Néron functions on a compact Riemann surface. These are with respect to a canonical volume form.

Let X be a complete nonsingular curve over the complex numbers C, so a compact Riemann surface. Let L be a metrized line sheaf with c^∞ metric ρ represented by the family $\{(U_i, \varphi_{ij}, \rho_i)\}$. Note that $\{\varphi_{ij}\}$ is the usual Čech 1-cocycle associated with the line sheaf, and for each i a function

$$\rho_i : U_i \rightarrow R_{\geq 0}$$

satisfies

$$\rho_i = |\varphi_{ij}|^2 \text{ on } U_i \cap U_j.$$

We define the Chern form of the metric to be

$$c_1 rho) = dd^c log \rho_i \ on \ U_i.$$

The Chern class $c_1(L)$ is defined to be the class of $c_1(\rho)$ in H_{dR^2}.

A form of type (1.1) is expressed locally at every pint in terms of a coordinate z as

$$f(z, \bar{z})\sqrt{-1} dz \wedge d\bar{z}$$

with f smooth. If f is real positive, then we say that the form is positive. By a volume form we mean a (1,1)-form which is everywhere positive. If φ is a real (1,1)-form, we say that φ is normalized if

$$\int_X \varphi = 1.$$

A metric ρ is called positive if the associated Chern form $c_1(\rho)$ is positive.

DEFINITION. *Let D be a divisor on a compact Riemann surface X. By a Green's function for D with respect to φ we mean a function*

$$g_D : X - supp(D) \to R$$

satisfying the following conditions:

GF1. If D is represented by (U,f) on an open set U, then there exists a c^∞ function α on U such that for all $P \notin supp(D)$, we have

$$g_D(P) = -log|f(P)|^2 + \alpha(P).$$

GF2. $dd^c g_D = (degD)\varphi$.

GF3. $\int_X g_D \varphi = 0$

We will see a Green's function exists with respect any smooth normalized (1,1)-real form on a compact Riemann surface X and as a function on $X \times X$ outside the diagonal, the Green's function is smooth.

One can define a Hilbert space structure on the space of differentials of first kind, denote dfk, by the hermitian product

$$(\varphi, \psi) \mapsto \frac{\sqrt{-1}}{2} \int_X \varphi \wedge \bar{\psi} = <\varphi, \psi>.$$

Let $\varphi_1, \cdots, \varphi_g$ be an orthonormal basis for dfk. Then

$$\mu = \frac{\sqrt{-1}}{2g}(\varphi_1 \wedge \bar{\varphi}_1 + \cdots + \varphi_g \wedge \bar{\varphi}_g)$$

is independent of the choice of orthonormal basis, and will be called the canonical 2 form or volume form on X. By definition of an orthonormal basis, we have

$$\int_X \mu = 1.$$

Let Δ be the diagonal on $X \times X$, which is a divisor on $X \times X$, and let P_1, P_2 be the projections on the first and second factor of $X \times X$ respectively. Let

$$\Phi = P_1^*\mu + P_2^*\mu - \frac{\sqrt{-1}}{2}\sum_{j=1}^{g}(P_1^*\varphi_j \wedge P_2^*\bar{\varphi}_j + P_2^*\varphi_j \wedge P_1^*\bar{\varphi}_j).$$

Then Φ is the Chern form of some metric on $O_{X\times X}(\Delta)$.

THEOREM 4.1. *Let ρ be a metric on $O_{X\times X}(\Delta)$ such that $c_1(\rho) = \Phi$. Let s be a holomorphic section of $O_{X\times X}(\Delta)$ such that $(s) = \Delta$, and let*

$$g = -log|s|_\rho^2.$$

After adding a constant to g (or equvalently changing s by a constant), then g is the Green's functionassociated with the form μ on X.

proof. See[7].

REMARK 4.2. *Let X be a nonsingular complete curve over the complex numbers. Let g_D be the Green's function associated with a divisor D. Then for the ordinary absolute value ν , the function*

$$\frac{1}{2}g_D$$

is a Weil function associated with D.

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