□論 文□

交叉路 分析을 위한 不連續 待機行列 模型 開發

A Discrete Time Queueing Model for Intersection Analysis

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CONTENTS

요 약

信號化된 交叉路의 運營效率을 測定하기 위해 현재 世界的으로 광범위하게 使用되는 척도는 交叉路 通過車輛의 平均遲滯時間이다.

그간 交叉路 分析을 위해 많은 대기행렬 模型이 발표되어 왔고 또 그중 일부가 현재 使用中에 있는데 이들은 모두 steady-state를 假定 한 解法이다. 그러나 steady-state 모형은 시간에 따른 대기행렬 길이의 變化를 고려하지 못하므로 現實的인 分析에 限界가 있는 方法論이다. 그러므로 正確한 交叉路 遲滯時間 算出을 위해서는 time-dependent한 分析模型의 개발이 要求된다.

本 硏究에서는 discrete Markov chain을 利用하여 단순히 單位時間 동안의 到着率과 出發率로써 transition probabilities를 計算하는 새로운 대기행렬 模型을 開發하였다. 開發된 不連續 대기행렬 模型을 利用하여 交叉路 分析을 할 경우 旣存의 交叉路 遲滯模型과 比較하여 期待되는 改善效果는 다음과 같다. 需要變化를 고려한 dynamic한 分析으로 現實的이고 正確한 豫測을 할 수 있다. 信號連動에 의한 영향을 分析할 수 있다. 그리고 獨立交叉路 뿐만 아니라 幹線道路, 나아가서 network 分析을 할 수 있으며, 동시에 주어진 交通與件에 대해 信號連動化를 위한 최적값을 산출해 낸다.

1. Introducition

Queueing theory was developed to provide models to predict behavior of systems that attempt to provide service for randomly arriving demands, and not unnaturally, then, the earliest problems studied were those of telephone traffic congestion. The pioneer investigator was the Danish mathematician A. K. Erlang, who, in 1909, published The Theory of Probabilities and Telephone Conversations[1]. After the Second World War when applications of mathematical models and methods in technology and other applied areas rose to a level previously unknown, it was realized that queueing theory too had a very broad field of applicability to various scientific and organizational phenomena[2].

There are extensive theoretical work on service systems, but they produce only steady-state results that are analytically simple. This is often done with little regard for the mathematical assumptions underlying these results. Moreover such results commonly do not answer the real questions one is facing in the design of a facility, and in rare cases the measures of performance based on steady-state assumptions may actually be misleading. For instance, for a system with rare arrivals of large groups, steady-state solution gives no information on the fluctuations of the queue length except an averaging property. Ignoring such fluctuations in a design may have catastrophic results. Especially for the analysis of system which has to deal with the demand of a certain time period, the steady-state assumption is ill-suited.

And, for the signalized intersection system, there are a number of steady-state delay models. However, most of existing models have deficiencies generally in expecting really consistent and accurate results. Any steady-state model that does not assume uniform arrivals will estimate that the delay approaches infinity as the v/c ratio approaches 1 which does not develop at real intersection.[3] Moreover, the estimated delay has to be adjusted by the progression factor for the central controlled signal system that is the general operation technique in today, which increases the inaccuracy resulting from the multiplication of two separate methods which have individually different assumptions.

Therefore it is strongly required that the methods to account for the way the queue varies over time should be developed, which means time dependent analysis. Unfortunately, the modeling of transient queue behavior is not a very well-developed field. There are very few results on the transient analysis, which are analytically explicit. In the continuous time queue analysis, the analytical expressions developed so far are for Poisson arrivals and exponential service times because the mathematics become extremely complicated with the slightest relaxation from those assumptions. And even the developed tend to be complicated and are nearly unsuitable for direct numerical analysis. So the results for the time-dependent behavior of its state probabilities leave much to be desired.

In this paper the discrete time approach to transient probabilities is introduced. In the transition probability, an approach using the number of arrivals and departures during the given discrete time interval is developed, which could shorten the computing time as well as be suitable for the delay estimation purpose.

2. Existing Methodology

2.1 Classic Continuous Time Solution

Several methods have been used to solve this problem since A. Clarke gave his time-dependent solution in 1953. Among them, it is Bailey's approach that has been the most popular over the years. Bailey's approach to the time dependent problem, proposed in 1954, was via generating functions for the partial differential equation. The state probabilities, $\{P_n(t)\}$, that at an arbitrary time t there are n customers, assuming that the initial system size at time zero is i, in a single-channel system with Poisson input, exponential service, and infinite waiting room is[4]

$$P_n(t)=e^{-(\lambda+\mu)t}[(\sqrt{\mu/\lambda})^{i-n} I_{n-i}(2\sqrt{\lambda\mu}t) + (\sqrt{\mu/\lambda})^{i-n+1} I_{n+i+1}(2\sqrt{\lambda\mu}t) + (1-\lambda/\mu)(\lambda/\mu)^n \sum_{k=n+i+2}^{\infty}(\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu}t)]$$

where $I_v(z) = \sum_{k=0}^{\infty}(z/2)^{v+2k}/k!(k+v)!$ is the modified Bessel function of the first kind of order k

i = initial system size at time 0

λ = arrival rate

μ = service rate.

2.2 Sharma's Solution

He proposed two dimensional state model for Markovian queues without reference to Bessel functions while the claassic M/M/1/∞ queue is obtained on the basis of a one-dimensional state model representing the number of units in the system at a given time. The two dimensions represent respectively the number of arrivals at, and the number of departures from, the system at a given time. It is remarked that the computation time of following expression is almost half of that of classic method. He showed that the computing time by the system Micro VAX II is taken 84.71 to 532.51 seconds according to parameter while classic method is taken 102.73 to 1221.59 seconds[2]. However, the expression is also complicated and the computing time is still unsatisfactory. The probabilities of customers remain in the system at time t which is taken form page 13 of [2] by putting $\rho=\lambda/\mu$ are

$$P(r,t)=(1-\lambda/\mu)(\lambda/\mu)^r+e^{-(\lambda/\mu)t}(\mu/\lambda)^r \sum_{k=0}^{\infty}(\lambda t)^k/k! \sum_{m=0}^{r+k}(k-m)(\mu t)^{m-1}/m!.$$

2.3 Neuts' Discrete Time Analysis

He proposed a relatively unsophisticated method which is useful in the analysis of unstable queues. In solving the unstable queues, the existing structural formulas are analytically complicated and to obtain the results from those continuous time solutions involve heavy algebraic manipulations. And, in analyzing it by the simulation techniques, the structure of the queue might be incorrectly or insufficiently used and excessive computing time would be required. So he tried to develop the discrete time queueing analysis method to obtain the numerical results efficiently by the Markov chain algorithm. He soved transient behavior of queues at each discrete time by computing in terms of the successive matrix convolution products of the transition matrix of the imbedded Markov renewal process. The probabilities of system size i and units of service time left j at nth unit time $P_n(i,j)$ for generally distributed service time system is[5]

(a) $P_n(0,0) = p_0[P_{n-1}(0,0) + P_{n-1}(1,1)]$,

(b) $P_n(i,j) = p_0P_{n-1}(i,j+1) + \sum_{v=1}^{i-1} p_{i-v}P_{n-1}(v,j+1) + r_j\{p_iP_{n-1}(0,0)+p_0P_{n-1}(i+1,1) + \sum_{v=1}^{i} p_{i-v+1}P_{n-1}(v,1)\}$,
for i=1,..,k and j=1,..,L_2−1.

(c) $P_n(i,j) = p_0P_{n-1}(i,j+1) + \sum_{v=i-k}^{i-1} p_{i-v}P_{n-1}(v,j+1) + r_j\{p_0P_{n-1}(i+1,1) + \sum_{v=i-k+1}^{i} p_{i-v+1}P_{n-1}(v,1)\}$,
for i=k+1,..,L_1−1 and j=1,..,L_2−1.

(d) $P_n(L_1,j) = P_{n-1}(L_1,j+1) + \sum_{v=1}^{k} (1-\sum_{k=0}^{v-1}p_k) P_{n-1}(L_1v, j+1) + r_j\{\sum_{v=1}^{k} (1-\sum p_k)P_{n-1}(L_1-v+1,1)\}$,
for j=1,..,L_2−1.

(e) $P_n(i,L_2) = r_{L2}\{p_iP_{n-1}(0,0) + p_0P_n(i+1,1) + \sum_{v=1}^{i} P_{n-1}(v,1)\}$, for i=1,..,k.

(f) $P_n(i,L_2) = r_{L2}\{p_0P_{n-1}(i+1,1) + \sum_{v=i-k+1}^{i} p_{i-v-1} P_{n-1}(v,1)\}$,
for i=k+1,..,L_1-1.

(g) $P_n(L_1,L_2) = r_{L2}\{\sum_{v=1}^{k} (1-\sum p_k)P_{n-1}(L_1-v+1,1)\}$,

where p_v = probability of v arrivals during a unit time
r_v = probability that a customer requires v units of service time
k = max number of arrivals during a unit time
L_1 = system capacity
L_2 = max units of service time.

3. A New Approach to Discrete Time Queue

3.1 Description of Model

The theory of queues, more generally that of stochastic models, suffer from the insufficient development of the interface between structural−analytical results on one hand and directly applicable numerical methods on the other hand. At this point of view, Neuts' discrete time method is very useful for the transient analysis.

The new model also basically forms the discrete time queueing analysis. Mathematically the model is built in terms of the

Markov chain and derives discrete time queue features from the transition probabilities of this chain and the single step transition probabilities can be arranged simply using the number of arrivals and departures during a unit time interval. So, in the new approach, any discrete probability arrival distribution with geometric service rate system can be assumed. However, so long as the transition probabilities satisfy the Markov chain, one may relaxed from the structural restrictions of probability distribution which are merely mathematical assumption for the interpretation of real system so that any empirical[1] probabilitic distributions can be employed in obtaining the transition probabilities.

The model in this study is much simpler than the classic continuous time solutions. At a glance, this approaches are to be regarded as approximations to continuous ones. It may usually be traced to the prevailing thought before the trial to computer. And, from the computational viewopint, continuous time models are substantially more delicate to analyze. Even in comparison with other discrete time methods which analyze the residual service time for each unit time interval, the new model treats the sequences of queues with more efficiency, so that it is suitable for system analysis that there is no necessity to analyze the waiting time distribution separately. Consequently, from the viewpoint of numerical analysis, this model has the most obvious advantage in computation time while the system may be evaluated accurately.

3.2 Discrete-Parameter Markov Chain

Let the sequence of random variables (arrivals and departures), $\{X_{i,j}=0,1,.. \mid X_j=0,1,..\}$, represent the exhaustive and mutually exclusive states of a system at any time. Then the markov chain is, for all j,

$$Pr\{X_j=n \mid X_0=k_0, X_1=k_1, X_2=k_2,...,X_{j-1}=k_{j-1}\}$$
$$= Pr\{X_j=n \mid X_{j-1}=k\}$$

Consider the intersection queueing system, where the number of arrivals and departures during a unit time interval T are independently and identically distributed random variables with the discrete probability distributions. Then the imbedded stocahstic process $X(t_i)$, where X denotes the number of vehicles in the system and $t_1,t_2,t_3,...$ are the successive unit times, can be shown to be Markov chain as follows. Since the state space is discrete, let us use a subscript notation so that X_j represents the number of vehicles remaining in the intersection system at j th unit time. We can then write for all $n \geq 0$ that

$$X_{j+1} = \begin{cases} X_j + A_{j+1} - D_{j+1} \ (X_j \geq 1) \\ A_{j+1} \ (X_j = 0) \end{cases}$$

where X_j is the number in the system at j th unit time and A_j+1 and D_j+1 are the number of vehicles arrived and departed respectively during the unit time interval T_j+1.

The random variable A_j+1 and D_j+1 by assumption depend only on T_j+1 and neither on the previous number of arrivals and

1) Empirical means relying on observation.

departures nor on the length of the queue, so let it henceforth be denoted by A and D. Since the unit time T_j+1 has the identical time interval for all times, so let it henceforth be denoted by T. And let

an = Pr{n arrivals during T}

dn = Pr{n departures during T}.

Then it follows that

for n=0,

$$\Pr\{X_{j+1}=0 \mid X_j=k\} = \begin{cases} a_0 & (k=0) \\ a_0 d_k & (k\geq 1), \end{cases}$$

for n≥1,

$$\Pr\{X_{j+1}=n \mid X_j=k\} = \begin{cases} a_n & (k=0) \\ \sum_{r=n-k}^{n} a_r d_{k+r-n} & (1\leq k<n) \\ \sum_{r=0}^{n} a_r d_{k+r-n} & (n\leq k). \end{cases}$$

Hence we can see that the imbedded process is Markovian, since only the indices(k,n,r) are involved and furthermore since the state variable is discrete, it is discrete-parameter Markow chain.

Now let the state probabilities $P_n(j) = \Pr\{X_j=n\}$ and the conditional probabilities $\Pr\{X_j=n \mid X_{j-1}=k\} = q_{kn}$, then

$$P_n(j) = \sum_k P_r\{X_j=n, X_{j-1}=k\}$$
$$= \sum_k P_r\{X_j=n \mid X_{j-1}=k\} \Pr\{X_{j-1}=k\}$$
$$= \sum_k P_k(j-1) q_{kn}.$$

For homogeneous chains,

$$P_n(j) = \sum_i P_r\{X_j=n, X_0=i\} \Pr\{X_0=i\}$$
$$= \sum_i P_i(0) q_{in}^{(j)}.$$

$$\text{And letting } \vec{P}(j) = \begin{pmatrix} P_0(j) \\ P_1(j) \\ P_2(j) \\ \vdots \end{pmatrix},$$

$$\vec{P}(j) = \vec{P}(0) P^{(j)}.$$

And the transition matrix is

$$P = \begin{pmatrix} q_{00} & q_{01} & q_{02} & q_{03} & \cdots \\ q_{10} & q_{11} & q_{12} & q_{13} & \cdots \\ q_{20} & q_{21} & q_{22} & q_{23} & \cdots \\ q_{30} & q_{31} & q_{32} & q_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where $\sum_n q_{kn} = 1$ for all k,

$q_{kn} \geq 0$ for all k and n.

3.3 Geometric Arrivals and Departures

Consider a single server queueing system with the finite capacity N in discrete time Markov chain. The elementary time interval is chosen as our time unit. And assume that the

number of arrivals and departures during the successive unit time intervals are independent, identically distributed random variables. Furthermore' ar, r=0,1,...,m, is the probabilities that r vehicles join the system during a given unit time interval. ($a_0+a_1+....+a_m=1$). And d_s, s=0,1,...,n, is the probabilities that s vehicles depart the system during a given unit time interval($d_0+d_1+....+d_n=1$). Then the transition probability will be different in compliance with the assumptions of arrival and departure distribution.

As mentioned earlier in 3.1, the new approach, so long as the transition probabilities satisfy the Markow chain, could be applied to any kind of discrete probability distribu tions. However, one should be aware that, even though it is a discrete probability distribution, there will be errors if Pr{more than 1 departure during a unit time interval T} is not zero. Since, in those service systems, there is such an implicit assumption in the transition probability q_{ij} that the number of departures during one unit time interval does not exceed i even though the maximum number of departures is n. So there are errors in transition probability q_{ij} when $i<s$ and $0<i$. However, there are no such errors in geometric service models. Since m is 1 in geometric distributions so the case of $i<s$ is only when i is zero. In this study, the model assumes the geometric arrival and departure distribution. In geometric arrival and departure assumprions, m and n are 1 so the probabilities of arrivals and departures for the single transition time are

$a_0 = (1-\lambda)T$

$a_1 = \lambda T$

$d_0 = (1-\mu)T$

$d_1 = \mu T$

where T is the unit time interval.

3.4 State Probabilities

The state probabilities $P_n(j)$ in geometric arrival departure assumption is

Then $P_n(j)$ satisfy the following recurrence relations in j for all $j \geq 0$:

For $n=0$,

$P_0(j) = P_0(j-1)\{a_0\} + P_1(j-1)\{a_0 d_1\}$.

For $n=1$,

$P_1(j) = P_0(j-1)\{a_1\} + P_1(j-1)\{a_0 d_0 + a_1 d_1\} + P_2(j-1)\{a_0 d_1\}$.

For $2 \leq n < N$,

$P_n(j) = P_{n-1}(j-1)\{a_1 d_0\} + P_n(j-1)\{a_0 d_0 + a_1 d_1\} + P_{n+1}(j-1)\{a_0 d_1\}$.

For $n=N$,

$P_n(j) = P_{N-1}(j-1)\{a_1 d_0\} + P_N(j-1)\{a_0 d_0 + a_1 d_1 + a_1 d_0\}$.

And the transition matrix is

	0	1	2	3	4	5		N-3	N-2	N-1	N
0	a_0	a_1	0	0	0	0	.....	0	0	0	0
1	a_0d_1	$a_0d_0+a_1d_1$	a_1d_0	0	0	0	.....	0	0	0	0
2	0	a_0d_1	$a_0d_0+a_1d_1$	a_1d_0	0	0	.....	0	0	0	0
3	0	0	a_0d_1	$a_0d_0+a_1d_1$	a_1d_0	0	.....	0	0	0	0
.	.	.	.	.	.	.		.	.	.	.
.	.	.	.	.	.	.		.	.	.	.
.	.	.	.	.	.	.		.	.	.	.
N-1	0	0	0	0	0	0	.....	0	a_0d_1	$a_0d_0+a_1d_1$	a_1d_0
N	0	0	0	0	0	0	.....	0	0	a_0d_1	$a_0d_0+a_1d_1+a_1d_0$

3.5 Intersection Delay Model

Suppose a signalized intersection system which has two different demand rates as illustrated in Figure 3.1. In this case, there are four different time intervals by the combination of arrival and service rates. Since $P = f\{\lambda,\mu\}$, each interval has the different transition marix. Using the probabilities at the end of each interval as the initial probabilities for the next, the state probabilities at unit time j is

$$\vec{P}(j) = \vec{P}(0)\,\mathbf{P}^{(j1)}\,\mathbf{P}_2^{(j2)}\,\mathbf{P}_3^{(j3)}\,\mathbf{P}_4^{(j4)}.$$

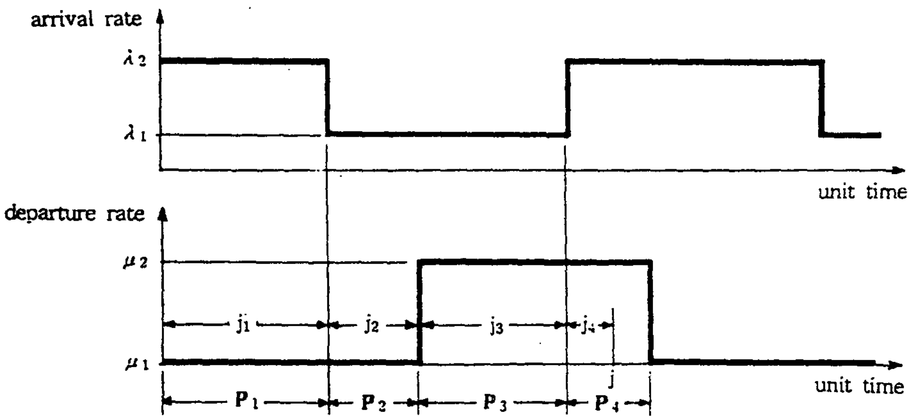

Fig 3.1 Arrivals and departures of intersection system

For accuracy, one can employ more variable, or non−stationary arrival rates. Moreover, more detailed departure rates, if available, are to be applied to the model. The recurrence relations are valid also for the system in which the demand probabilities vary with time can be done with minor obvious changes but the point of issue for the detailed variable demand rates is the computing time. For the non−stationary arrivals, the demand interval is divided into short time period assuming that the mean demand flow during T is considered to be stationary[1)].

When the demand interval is equal to the unit time interval T then the above Equation will be transformed to

$$\vec{P}(j) = \vec{P}(0)\ \mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3\ \ldots\ldots\ \mathbf{P}_j.$$

Now let Q(j) be the random variable representing the queue length at unit time j, then

$$Q(j) = \vec{P}(j)\ \vec{V}^T.$$

And the total delay of a signal cycle D is

$$D = T\sum_{j=1}^{CY/T} Q(j).$$

Letting the unit delay(actually this not the delay) $T^*Q(j) = D(j)$,

1) The demand interval should be greater than or equal to the unit time interval since it is assumed, in Markov chain, that the transition probabilities are stationary over time.

$$D = \sum_{j=1}^{CY/T} D(j).$$

And the average delay per vehicle d is

$$d = \sum_{j=1}^{CY/T} \{D(j)/\lambda(j)\}.$$

4. Conclusions

In this study, a new approach to transition probabilities is developed by simply using the number of arrivals and departures during a unit time interval without analysis of service time. In delay estimation, computation time is very important particularly for wide network analyses. From the viewpoint of numerical analysis, the new approach for the transient behavior of intersection queue has the most obvious advantage in computation time compared with not only classic continuous time solution but also existimg discrete time analysis while the system may be evaluated accurately.

Mathematically, the model is built in terms of the Markov chain and derives discrete time queue features from the transition probabilties of this chain. Then, by this discrete time analysis method, a signalized intersection queueing system with time-varying arrivals is designed with consideration of the progression effect.

The mathematical exression and its computation is more sophisticated than those of existing models. And the information required are also more demanding. To expect the reliable results, more detailed information in the traffic pattern are reguired. An improvement should be followed in obtaining the precise information and in manipulating them efficiently. However, considering the more various factors affecting delay such as time-varying demand, system capacity, offset value, etc., the estimated results form the model would be reliable.

The expected improvements by the proposed model are:

- dynamic analysis
- realistic estimation particularly when v approaches c
- accurate analysis with time varying arrival rates
- flexible selection of analysis period
- consideration of progression quality in the model
- analysis of progression effect on delay when v/c > 1
- used not only for single intersections but also for networks
- estimation of queue growth rate and queue clearance time.

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