STRONG UNIQUE CONTINUATION
OF THE SCHRÖDINGER OPERATOR

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1. Introduction

It is well known that if $P(x, D)$ is an elliptic differential operator, with real analytic coefficients, and $P(x, D)u = 0$ in an open, connected subset $\Omega \in \mathbb{R}^n$, then $u$ is real analytic in $\Omega$. Hence, if there exists $x_0 \in \Omega$ such that $u$ vanishes of $\infty$ order at $x_0$, $u$ must be identically 0. If a differential operator $P(x, D)$ has the above property, we say that $P(x, D)$ has the strong unique continuation property (s.u.c.p.). If, on the other hand, $P(x, D)u = 0$ in $\Omega$, and $u = 0$ in $\Omega'$, an open subset of $\Omega$, implies that $u = 0$ in $\Omega$ we say that $P(x, D)$ has the unique continuation property (u.c.p.). Finally, if $P(x, D)u = 0$ in $\Omega$, and $\text{supp} u \subset K \subset \Omega$ implies that $u = 0$ in $\Omega$, we sat that $P(x, D)$ has the weak unique continuation property (w.u.c.p.).

The first results in this direction are to be found in the work of T. Carleman[2] in 1939. He was able to show that $P(x, D) = \Delta + V(x)$ in $\mathbb{R}^2$ has the (s.u.c.p.) whenever the function $V(x)$ is in $L_{\text{loc}}^\infty(\mathbb{R}^2)$. In order to prove this result he introduced a method, the so called Carleman estimates, which has permeated almost all subsequent work in this subject. In this context, a Carleman estimate is, roughly speaking, an inequality of the form:

$$||e^{t\phi}f||_{L^2(U)} \leq C||e^{t\phi}\Delta f||_{L^2(U)}$$

for all $f \in C_0^\infty(U)$, $U$ an open subset of $\mathbb{R}^2$, and a suitable function $\phi$, where the constant $C$ is independent of $t$, for a sequence of real values of $t$ tending to $\infty$. There are many papers on which this work is based. For a reference, see the survey by C.E. Kenig[6].

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Theorem. Let $U$ be a non-empty connected open subset of $\mathbb{R}^n$, and $u$ be a solution of the differential equation

$$(\Delta + \nabla W)u = 0$$

(1)

Here $\Delta$ is the Laplace operator, $W \in L^r_{\text{loc}}(\mathbb{R}^n)$ for some suitable $r$. If $u$ vanishes at an open subset of $u$, then $u = 0$ identically on $U$.

This kind of unique continuation theorem for the Schrödinger operator $\Delta + V$ when $V \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$ has been studied by many mathematicians.([2][3][4][5])

In [4] D. Jerison showed (s.u.c.p.) holds for the operator $\Delta + V$, where $V \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$. He also suggested unique continuation hold for operators of the form $\Delta + V$, where $V = \sum \partial V_j / \partial x_j$, and $V_j \in L^r_{\text{loc}}(\mathbb{R}^n)$, $r = (3n - 2)/2$.

This last hypothesis on the potential is closer in spirit to the condition on potentials advanced by B.Simon[8].

In [7], unique continuation of the differential equation

$$(\Delta + \sum a_j \partial / \partial x_j + b)u = 0$$

(2)

with $a_j \in L^r_{\text{loc}}(\mathbb{R}^n)$, $b \in L^r_{\text{loc}}(\mathbb{R}^n)$ was shown.

To prove unique continuation for (1), we need the following Carleman inequality proved by the author[4].

$$||e^{ts(y)} \nabla f||_{L^{q_1}(U \setminus \{0\})} \leq C ||e^{ts(y)} \Delta f||_{L^p(U \setminus \{0\})}.$$  

(3)

where $1/p - 1/q_1 = 1/r$ for $C$ independent of $t$ as $t$ goes to infinity, $f \in C^\infty_0(U \setminus \{0\})$, and $s(y)$ is a suitable weight function which is radial and radially decreasing. The key feature that distinguishes this inequality from ordinary Sobolev inequalities is that the constant $C$ is independent of the parameter $t$.  

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Since these are Sobolev inequalities, exponent $s = n/2$ is the natural one we can expect. But we need some restriction for $r$. In our case the largest value we can expect for $r$ is $r = (3n - 2)/2$, and we will choose $p = (6n - 4)/(3n - 2)$. Then $q_1 = 2$, $q_2 = (6n - 4)/(3n - 6)$.

Furthermore, we will use the weight function $s(y)$ defined implicitly by $y = -s(y) + e^{-c s(y)}$ when $y = \log|x| < 0$. Roughly speaking, $e^{t s(y)} \sim |x|^{-t}$. The idea is from Alinhac - Baouendi [1] and has been used by Hörmander[3].

**Notations**

1.**The Dirac operator** is a first-order constant coefficient operator on $R^n$ of the form $D = \sum_{j=1}^{n} \alpha_j \partial / \partial x_j$, where $\alpha_1, \ldots \alpha_n$ are skew hermitian matrices satisfying the Clifford relations: $\alpha_j^* = -\alpha_j$ and $\alpha_j \alpha_k + \alpha_k \alpha_j = -2 \delta_{jk}; j, k = 1, \ldots, n$. Also $D^2 = -\Delta$.

2.**Polar coordinates**

Let $S$ denote the unit sphere in $R^n$. For $y \in R$, and $w \in S$, $x = e^y w$ gives polar coordinates on $R^n$, i.e., $y = \log|x|$ and $w = x/|x|$. The operator

$$L = \sum_{j < k} \alpha_j \alpha_k (x_j \partial / \partial x_k - x_k \partial / \partial x_j)$$

acts only in the $w$-variables—$[L, \partial / \partial y] = 0$. We will view $L$ as an operator on the sphere $S$. Let

$$\hat{\alpha} = \sum_{j=1}^{n} \alpha_j x_j / |x|, \text{ then}$$

$$\hat{\alpha} D = e^{-y} (\partial / \partial y - L);$$

and since $\hat{\alpha}^2 = -1$,

$$e^y D = \hat{\alpha} (\partial / \partial y - L)$$

(5)
Note that $$\Delta = e^{-2y}(\partial^2/\partial y^2 + (n-2)\partial/\partial y + \Delta_S),$$

where $$\Delta_S$$ denotes the Laplace-Beltrami operator of the sphere. It follows from $$D^* = D, \quad D^2 = -\Delta$$ that

$$L(L + n - 2) = -\Delta_S$$

In general if $$\psi \in C^\infty(R),$$ then (6) implies that in polar coordinates $$x = e^y w,$$

$$e^{ts(y)} e^y D e^{-ts(y)} h = \alpha A_t h$$

where $$A_t = \partial/\partial y - (ts(y) + L).$$

Now we will prove the theorem.

**Proof.** Choose a small subset $$A \subset U$$ which will be decided later. From (3) and (4) we have

$$\|e^{ts(y)} \nabla u\|_{L^2(A)} + \|e^{ts(y)} u\|_{L^q_2(A)}$$

$$\leq C\|e^{ts(y)} \Delta u\|_{L^p(R^n \setminus A)} + C\|e^{ts(y)} \Delta u\|_{L^p(A)}$$

On the other hand, from (1), the right hand side is bounded by

$$C\|e^{ts(y)} \Delta u\|_{L^p(R^n \setminus A)} + C\|e^{ts(y)} (\nabla W) u\|_{L^p(A)}$$

Integrating by parts, we find the second part of the above is bounded by

$$C\|e^{ts(y)} W(\nabla u)\|_{L^p(A)} + C\|e^{ts(y)} (\nabla W) u\|_{L^p(A)}$$

Hölder’s inequality tells us the above is bounded by

$$C\|e^{ts(y)} (\nabla u)\|_{L^2(A)} \|W\|_{L^r(A)} + C\|ts'(y)e^{ts(y)} u\|_{L^2(A)} \|W\|_{L^r(A)}$$

If we sum all the terms we finally obtain

$$\|e^{ts(y)} \nabla u\|_{L^2(A)} + \|e^{ts(y)} u\|_{L^q_2(A)}$$
Strong unique continuation of the Schrödinger operator

\[ \leq C ||e^{ts(y)} \Delta u||_{L^p(R^n \setminus A)} + C ||e^{ts(y)}(\nabla u)||_{L^2(A)} ||W||_{L^r(A)} \\
+ C ||t s'(y) e^{ts(y)} u||_{L^2(A)} ||W||_{L^r(A)} \]

The idea is to make cancellations of the last two terms on the right against the left hand side. If we choose \( A \) small as possible as \( ||W||_{L^r(A)} < 1/4 \), and use the following \( L^2 \to L^2 \) estimate

\[ t ||e^{ts(y)} u||_{L^2(A)} \leq C ||e^{ts(y)} \nabla u||_{L^2(A)} \]

(10)

After cancellations, we obtain the following:

\[ ||e^{ts(y)} \nabla u||_{L^2(A)} + ||e^{ts(y)} u||_{L^{q_2}(A)} \]

\[ \leq C ||e^{ts(y)} \Delta u||_{L^p(R^n \setminus A)} \]

Since \( s(y) \) is radial and decreasing, choose a value \( a \in \partial A \). Then

\[ ||e^{ta} \nabla u||_{L^2(A)} + ||e^{ta} u||_{L^{q_2}(A)} \leq ||e^{ts(y)} \nabla u||_{L^2(A)} + ||e^{ts(y)} u||_{L^{q_2}(A)} \]

\[ \leq C ||e^{ta} \Delta u||_{L^1(R^n \setminus A)} \]

\[ \leq C' \]

Letting \( t \) to infinity, we are forced to \( u = 0 \) identically in \( A \).

References

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