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RELATIVELY PATH–CONNECTED ORTHOMODULAR LATTICES

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1. Preliminaries

Every irreducible block-finite orthomodular lattice is simple [9] and every irreducible orthomodular lattice such that no proper p-ideal of L contains infinitely many commutators is simple [5]. Every finite (height) OML L which does not belong to the variety generated by $MO2$ has one of the OML $MO3$, $2^3 \circ 2^2$, D_{16}, $OMLHOUSE$ as the homomorphic image of a subalgebra of L [3]. In this paper, we extend these results.

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [7]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in B$.

We write $M \leq L$ if and only if M is a subalgebra of an OML L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the *relative orthocomplementation* $\sharp$ on $M[a, b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a, b]$. In particular, we write $[a, b]$ instead of $L[a, b]$ if there is no ambiguity. Whenever we refer to an interval $[a, b]$ of an OML, we shall consider it as being an OML with the relative orthocomplementation.

The *commutator of a and b* of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. For elements a, b of an OML, we say *a commutes with b*, in symbols $a \,\mathbf{C}\, b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \mid x \,\mathbf{C}\, m \quad \forall m \in M\}$ is called the *commutant* of M in L. The set $\mathbf{C}(L)$ is called the *center*

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of L and $\mathbf{C}(L) = \bigcap\{\mathbf{C}(a) \mid a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0, 1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathcal{A}_L$. Note that $\bigcup \mathcal{A}_L = L$ and $\bigcap \mathcal{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if and only if $|\mathcal{A}_L|$ is finite.

For any e in an OML L, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the *(principal) section generated by* e. Note that if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

DEFINITION (1.1). For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *weak path* in L is a finite sequence B_0, B_1, ..., B_n $(n \geq 0)$ in $\mathcal{A}_L$ satisfying $B_i \overset{wk}{\sim} B_{i+1}$ whenever $0 \leq i < n$. A *path* in L is a finite sequence B_0, B_1, ..., $B_n (n \geq 0)$ in $\mathcal{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The (weak) path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. The *distance* $d(A, B)$ of blocks A, $B \in \mathcal{A}_L$ is defined to be the minimum of the lengths of all strictly proper paths joining A and B if such a path exists and to be ∞ if there is no strictly proper path joining A and B [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. We call e a *vertex* of L. In fact, e is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. We say that A and B are *linked* (*strongly linked*) if $A \sim B$ $(A \approx B)$ *at* e and use the notation $A \sim_e B$ $(A \approx_e B)$. Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section* S_e" to describe $A \overset{wk}{\sim} B$ [4].

DEFINITION (1.2). Let L be an OML, and $A, B \in \mathcal{A}_L$. We will say that A and B are *weakly path-connected (in L)*, *path-connected (in L)*, *strictly path-connected (in L)* if A and B are joined by a weak path, a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *weakly path-connected (in L)*, *path-connected (in L)*, *strictly path-connected (in L)* if any two blocks in L are joined by a weak path, a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* iff each $[0, x]$ is path-connected for all $x \in L$.

We find some examples and properties of path-connected OMLs in [8]. In particular, every OML such that the height of the join-semilattice generated by the all commutators of L is finite is path-connected [8], which properly contains the following two theorems: every block-finite OML is path-connected [1]; every commutator-finite OML is path-connected [2].

LEMMA (1.3)[BRUNS]. *If L_1, L_2 are OMLs, $L = L_1 \times L_2$, $A, B \in \mathcal{A}_{L_1}$ and $C, D \in \mathcal{A}_{L_2}$, then $A \times C \sim B \times D$ holds in L if and only if either $A = B$ and $C \sim D$ or $A \sim B$ and $C = D$. If A and B are linked at a then $A \times C$ and $B \times C$ are linked at $(a, 0)$. If C and D are linked at c then $A \times C$ and $A \times D$ are linked at $(0, c)$ [1].*

LEMMA (1.4)[BRUNS]. *Let L_1, L_2 be OMLs and $L = L_1 \times L_2$. If L_1 and L_2 are non-Boolean, if any two blocks in L_1 can be joined by a path of length at most m, and if any two blocks in L_2 can be joined by a path of length at most n, then any two blocks in L can be joined by a strictly proper path of length at most $m + n$. If one of L_1, L_2 is Boolean, then any two blocks in the other can be joined by a path (proper path, strictly proper path) if and only if any two blocks in L have this property [1].*

PROPOSITION (1.5). *Let L be an OML, and $x \in L$. Then $\mathbf{C}(x)$ is path-connected if and only if $[0, x]$ and $[0, x']$ are path-connected.*

Proof. We know that $\mathbf{C}(x) = [0, x] \oplus [0, x']$. First, if $[0, x]$ and $[0, x']$ are path-connected, then $\mathbf{C}(x)$ is path-connected by lemma (1.4). Conversely, assume that $\mathbf{C}(x)$ is path-connected and let us prove that $[0, x]$

and $[0, x']$ are path-connected. It is sufficient to show that $[0, x]$ is path-connected by symmetry. Let A, B be distinct blocks in $[0, x]$ and let $D \in \mathcal{A}_{[0,x']}$. We may assume that $A \cup B \not\leq [0, x]$, otherwise A and B are path-connected in $[0, x]$. Then $A \oplus D$ and $B \oplus D$ are blocks in $\mathbf{C}(x)$ and hence path-connected in $\mathbf{C}(x)$. Let $A \oplus D = C_0 \oplus E_0 \sim C_1 \oplus E_1 \sim \ldots \sim C_n \oplus E_n = B \oplus D$ $(n \geq 2)$ be a path joining $A \oplus D$ and $B \oplus D$ in $\mathbf{C}(x)$ where $C_i \in \mathcal{A}_{[0,x]}$ and $E_i \in \mathcal{A}_{[0,x']}$ $\forall (0 \leq i \leq n)$. Then the sequence $C_0, C_1, \ldots, C_n$ satisfies $C_i \sim C_{i+1}$ in $[0. x]$ or $C_i = C_{i+1}$ by lemma (1.3). Let $M = \{i | C_i \sim C_{i+1},\ 1 \leq i \leq n-1\}$. Then $A = C_0 \sim C_{i_1} \sim \ldots \sim C_{i_k} \sim C_n = B$ where $i_j \in M$ such that $0 = i_0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n-1$. Thus A and B are path-connected in $[0, x]$, and hence $[0, x]$ is path-connected. We are done. ∎

PROPOSITION (1.6). *Every finite direct product of path-connected OMLs is path-connected.*

Proof. The simple induction proof is omitted. ∎

PROPOSITION (1.7). *Let L be an OML. Then the following are equivalent:*

(1) L is relatively path-connected;
(2) $\mathbf{C}(x)$ is path-connected $\forall x \in L$;
(3) S_x is path-connected $\forall x \in L$.

Proof. First, (1) iff (2) by proposition (1.5). (1) iff (3) by lemma (1.4) since $S_x = [0, x'] \oplus \{0, x\}$ and $\{0, x\} \cong 2^1$ is Boolean . ∎

2. Relatively Path-connected Orthomodular Lattices

Every path-connected has the following two properties. It is not known whether there is an OML for which the conclusion of (2.1) fails.

LEMMA (2.1). *Let L be a path-connected OML, and $x \in L \setminus \mathbf{C}(L)$. Then there exist two blocks $B, C \in \mathcal{A}_L$ such that $x \in B \setminus C$, and $B \cup C \leq L$*

Proof. Let $\mathbf{C}(x)$ be the commutant of $x \in L$, and $\mathcal{A}_{\mathbf{C}(x)}$ be all blocks in the subalgebra $\mathbf{C}(x)$ of L. Then $\mathcal{A}_{\mathbf{C}(x)}$ is properly contained in $\mathcal{A}_L$ since $x \notin \mathbf{C}(L)$. Thus there exist two blocks D, E such that $D \in \mathcal{A}_{\mathbf{C}(x)}$

and $E \in \mathcal{A}_L \backslash \mathcal{A}_{\mathbf{C}(x)}$. There exists a proper path $\{(B_j)\}_{j=0}^n$ from $D = B_0$ to $E = B_n$ since L is path-connected. Let k be the minimal index such that $B_k \notin \mathcal{A}_{\mathbf{C}(x)}$. Then $B_{k-1} \in \mathcal{A}_{\mathbf{C}(x)}$. Let $B_{k-1} = B$ and $B_k = C$. Then we are done. ∎

LEMMA (2.2). *Let L be a path-connected OML, and $A, B \in \mathcal{A}_L$. If $A \cap B \neq \mathbf{C}(L)$, then A and B are strictly path-connected.*

Proof. If one of the proper paths from A to B has length $n \geq 2$, it is a strictly proper path. Otherwise, it is a strictly proper path since $A \cap B \neq \mathbf{C}(L)$. ∎

Bruns has proved the following lemma (2.3) [1].

LEMMA (2.3)[BRUNS]. *Let L_1, L_2 be OMLs, $A, B \in \mathcal{A}_{L_1}$ and $C, D \in \mathcal{A}_{L_2}$ and $A \times C \sim_e B \times C \sim_f B \times D$. Then this path is strictly proper and there exists a unique block $F \in \mathcal{A}_{L_1 \times L_2}$, namely $F = A \times D$, such that $A \times C \sim_f F \sim_e B \times D$.*

The following lemma [6] is an extention of Bruns result [1].

LEMMA (2.4). *Let L be an OML, A, B, $C \in \mathcal{A}_L$ with $A \sim_e B \sim_f C$ and $e \perp f$. Then there exists a unique block D such that $A \sim_f D \sim_e C$. Moreover $D \neq B$.*

Proof. Let $A \sim_e B \sim_f C$ with $e \perp f$. Since $A \neq B$ and $B \neq C$ and $e \leq f'$ we have $0 < e \leq f' < 1$. Moreover, $f' \in B \cap S_e \subseteq S_e \cap (A \cup B) = A \cap B \subseteq A$, so $f \in A \cap B \cap C$. By symmetry, $e \in A \cap B \cap C$ also, so that $A, B, C \in \mathcal{A}_{\mathbf{C}(e,f)}$. Note that $\mathbf{C}(e, f) = L[0, e] \oplus L[0, f] \oplus L[0, e' \wedge f']$. Thus $A = A[0, e] \oplus A[0, f] \oplus A[0, e' \wedge f']$, $B = B[0, e] \oplus B[0, f] \oplus B[0, e' \wedge f']$, and $C = C[0, e] \oplus C[0, f] \oplus C[0, e' \wedge f']$. But $A[0, e'] = B[0, e']$ and $B[0, f'] = C[0, f']$ so that $A[0, e' \wedge f'] = B[0, e' \wedge f'] = C[0, e' \wedge f']$. Thus$A[0, e \vee f] \sim B[0, e \vee f] \sim C[0, e \vee f]$ by lemma (1.3). Hence there exists a unique $\widetilde{D} \in \mathcal{A}_{L[0, e \vee f]}$ with $A[0, e \vee f] \sim_f \widetilde{D} \sim_e C[0, e \vee f]$ by lemma (2.3). Moreover $\widetilde{D} \neq B[0, e \vee f]$. Let $D = \widetilde{D} \oplus A[0, e' \wedge f']$. Then the uniqueness may be stated with respect to L and $D \neq B$. We are done. ∎

DEFINITION (2.5). Two elements a, b of an OML L are said to be *perspective* if there exists $z \in L$ such that $a \vee z = b \vee z = 1$ and

$a \wedge z = b \wedge z = 0$. A *p-ideal* in L is a lattice ideal which is closed under perspectivity.

The lattice of all p-ideals in an OML L, $\mathcal{I}_P(L)$, is isomorphic with the *congruence lattice* of L [7]. Thus L is *simple* if and only if $\mathcal{I}_P(L)$ is a chain with two elements.

The following lemma was proved by Roddy [9].

LEMMA (2.6)[RODDY]. *Let L be an OML, $B, C \in \mathcal{A}_L$, $B \sim_d C$ and $I \in \mathcal{I}_P(L)$ with $I \cap B \not\subseteq [0, d']$. Then $d \in I$.*

The following three lemmas are generalizations of Roddy, and only minor modifications of Roddy proofs are required [9].

LEMMA (2.7). *Let L be a path-connected OML, $A, B, C \in \mathcal{A}_L$ with $B \sim_d C$ and $b \in \mathbf{C}(L) \cap [d, 1]$. Then there exists a path $B_0 \sim_{a_0} B_1 \sim_{a_1} \sim \dots \sim_{a_{n-1}} B_n$ satisfying: $B_0 = A$, $a_i \leq b$, $(0 \leq i < n)$ and $a_{n-1} = d$.*

Proof. There is a path from A to B since L is path-connected. Thus there exists a path with $B_0 = A$ and $a_{n-1} = d$ by joining a path from A to B and $B \sim_d C$. Let n be the minimum length of all such paths. Then $n \geq 2$. Suppose there is no path satisfying $a_i \leq b$, $(0 \leq i < n)$. Let l be the largest index with $a_l \not\leq b$ for all paths such that $B_0 \sim_{a_0} B_1 \sim \dots \sim_{a_{n-1}} B_n$ with $B_0 = A$ and $a_{n-1} = d$. Choose a path such that l is maximal. Then $l \leq n-2$. Suppose $l \neq n-2$. We have $a_{l+1} \leq b \leq a_l'$, since $b \in \mathbf{C}(L)$, $a_l \not\leq b$ and a_l is an atom of $B_l \cap B_{l+1}$. Thus there exists a block H such that $B_l \sim_{a_{l+1}} H \sim_{a_l} B_{l+2}$ by lemma (2.4). Thus there exists a path $B_0 \sim_{a_0} B_1 \sim \dots \sim_{l-1} B_l \sim_{a_{l+1}} H \sim_{a_l} B_{l+2} \dots \sim_{a_{n-1}} B_n$ contradicting the maximality of l. Hence $l = n-2$. Then there exist a block S with $B_{n-2} \sim_d S \sim_{a_{n-2}} B_n$ since $B_{n-2} \sim_{a_{n-2}} S \sim_d B_n$ with $d \leq b \leq a_{n-2}'$ by lemma (2.4) contradicting the maximality of n. We are done. ∎

LEMMA (2.8). *Let L be a relatively path-connected irreducible OML, $I \in \mathcal{I}_P(L)$, $B, C \in \mathcal{A}_L$ with $B \sim_d C$ and $0 \neq b' \in I \cap B \cap [0, d']$. Then there exist $D, E \in \mathcal{A}_L$ and $v \in V_L$ such that $D \sim_d E$, $b \in D$ and $v \in (I \cap D) \setminus [0, b']$.*

Proof. There exist A, $H \in \mathcal{A}_L$ with $A \sim H$ and $b \in A \setminus H$ by lemma (2.1) since $b \notin \mathbf{C}(L) = \{0, 1\}$. We have $H \sim_a A = B_0 \sim_{a_0} B_1 \sim \ldots \sim_d B_n$ with $a_i \leq b$ $(0 \leq i < n-1)$ in $\mathbf{C}(b)$ by lemma (2.7) since $\mathbf{C}(b)$ is path-connected. Let $v_0 = a$. If $v_i \in B_{i+1}$, then let $v_{i+1} = v_i$. If $v_i \notin B_{i+1}$, then let $v_{i+1} = a_i$. Then $v_0 = a \in I$ by lemma (2.6) since $b' \in A \setminus [0, a']$. If $v_{i+1} \neq v_i$, then $v_{i+1} = a_i$ and $v_i \notin B_{i+1}$. Thus $v_{i+1} \in I$. If $a \leq b'$, then $b' \in H$ contradicting $b \in A \setminus H$. Thus $a \not\leq b'$. Therefore $v_i \not\leq b'$ $\forall i$ $(0 \leq i < n-1)$ since $a \leq b$ implies $a_i \not\leq b'$. Let $v = v_{n-2}$, $B_{n-1} = D$ and $B_n = E$. We are done. ■

LEMMA (2.9). *Let L be a relatively path-connected irreducible OML such that no proper p-ideal contains infinitely many vertices. Let $B, C \in \mathcal{A}_L$ and $I \in \mathcal{I}_P(L)$ with $B \sim_d C$ and $I \cap B \neq \{0\}$. Then $d \in I$.*

Proof. We may assume that I is a proper p-ideal, and $I \cap B \subseteq [0, d']$ by lemma (2.6). Let $0 \neq b' \in I \cap B \cap [0, d']$. We construct a sequence as the following: Let $b'_0 = b'$, $B_0 = B$ and $C_0 = C$. If $b'_i \in [0, d']$, then $B_i \sim_d C_i$ and $0 \neq b'_i \in I \cap B_i \cap [0, d']$. There exists $D_i \sim_d E_i$ and $v_i \in V_L$ such that $D_i \sim_d E_i$, $b'_i \in D_i$ and $v_i \in (I \cap D_i) \setminus [0, b'_i]$, by lemma (2.8). Let $b'_{i+1} = v_i \vee b'_i$, $B_{i+1} = D_i$ and $C_{i+1} = E_i$. If $b'_i \notin [0, d']$, then let $b'_{i+1} = b'_i$, $B_{i+1} = B_i$ and $C_{i+1} = C_i$. Suppose $b'_i \in [0, d']$ for all $i \geq 1$. Then $\{b'_i\}$ is strictly increasing infinite sequence since $v_i \notin [0, b'_i]$. But $\{b'_i\} \subseteq \{b'_0 \vee (\bigvee V_0 | V_0 \subseteq (V_L \cap I)\}$ is finite since there is no proper p-ideal containing infinitely many vertices in L contradicting the sequence to be infinite. Thus there exists a smallest integer n such that $b'_n \notin [0, d']$. Hence $B_n \sim_d C_n$ and $b'_n \in (I \cap B_n) \setminus [0, d']$. Therefore $d \in I$ by lemma (2.6). ■

We are ready to prove the following theorem.

THEOREM (2.10). *If L is a relatively path-connected irreducible OML such that no proper p-ideal of L contains infinitely many vertices, then L is simple.*

Proof. Let $I \neq \{0\}$ be a p-ideal in L. Let $x \in I \setminus \{0\}$ and $d \in V_L$. Let $B \in \mathcal{A}_L$ with $x \in B$ and D, $E \in \mathcal{A}_L$ with $D \sim_d E$. Let $B = B_0 \sim_{a_0} B_1 \sim \ldots \sim_{a_{n-1}} B_n = D$ be a path joining B and D. Then $a_0 \in I$ by lemma (2.9). If $a_i \in I$, then $a_{i+1} \in I$ by lemma (2.9). Therefore $d \in I$

by lemma (2.9) since $a_{n-1} \in I$. Thus $V_L \subseteq I$. If $|V_L| = \infty$, then $I = L$ since there is no proper p-ideal of L containing infinitely many vertices. Therefore L is simple. If $|V_L| < \infty$, then $v = \bigvee V_L \in I$. Let A, $B \in \mathcal{A}_L$ with $v \in B$, and $A = B_0 \sim_{a_0} B_1 \sim \ldots \sim_{a_{n-1}} B_n = B$ a path from A to B. Then $v \in B_{n-1}$. If $v \in B_i$, then $v \in B_{i-1}$ by lemma (2.9). Thus $v \in A$. Hence $v \in \mathbf{C}(L)$. Then $v = 1$, since L is irreducible. Therefore $I = L$. We are done. ■

We need the following proposition to prove the structure theorem of a path-connected OML with a maximal Boolean factor.

PROPOSITION (2.11). *Every infinite direct product of OMLs containing infinitely many non-Boolean factors is a nonpath-connected OML.*

Proof. Let $L \cong \prod_{\alpha \in I} L_\alpha$ where $|I| \geq \omega$ and each L_α is OML. Let $J = \{j \in I \mid L_j$ is a Boolean algebra $\}$. Then $\mathbf{B} = \prod_{j \in J} L_j$ is a Boolean factor of L. Thus $L \cong \mathbf{B} \times \prod_{i \in I \setminus J} L_i$ such that $|I \setminus J| \geq \omega$ and each L_i $(i \in I \setminus J)$ is non Boolean path-connected OML. Therefore it is sufficient to show that $\prod_{i \in I \setminus J} L_i$ is not path-connected by lemma (1.4). Since each L_i $(i \in I \setminus J)$ is a path-connected OML containing at least two distinct blocks, there exist distinct $A_i, B_i \in \mathcal{A}_{L_i}$ $\forall i \in I \setminus J$ such that $A_i \cup B_i \leq L_i$. Let $A = \prod_{i \in (I \setminus J)} A_i$ and $B = \prod_{i \in (I \setminus J)} B_i$. Then A and B are not path-connected since there is no path of finite length from A to B by lemma (1.3). ■

THEOREM (2.12). *If L is a path-connected OML with a maximal Boolean factor B, then $L \cong B \times L_1 \times L_2 \times \ldots \times L_n (n \geq 0)$, where $L_i (1 \leq i \leq n)$ are irreducible nonBoolean path-connected OMLs.*

Proof. We may assume that L is nonBoolean. Thus $L \cong B \times L_0$ where L_0 is a path-connected OML which has no nontrivial Boolean factor. If L_0 is irreducible, then there is nothing to prove. Thus we may assume that L_0 is reducible. Then L_0 has only finitely many irreducible nonBoolean path-connected factors, otherwise L_0 is not path-connected by proposition (2.11). ■

DEFINITION (2.13). An element e of an OML L is called a ***hyperatom*** if and only if every maximal chain in the interval $[0, e]$ has exactly three elements. L is called *homogeneous* if and only if e is a hyperatom of L whenever two blocks B and C meet in the section S_e.

An OML L is called the *horizontal sum* of a family $(L_i)_{i\in I}$ (denoted by $\circ(L_i)_{i\in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0,1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:

(1) if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \vee y = 1$;
(2) every block of L belongs to some L_i;
(3) if S_i is a subalgebra of L_i, then $\bigcup S_i$ is a subalgebra of L [2].

Bruns and Kalmbach have proved the following two lemmas and theorem (2.16) [3].

LEMMA (2.14). *Let L be an OML which does not belong to the variety $[MO2]$ generated by $MO2$, and satisfies $n(L) \leq 2^3$ where $n(L)$ is the maximum of the cardinal numbers $|B|$ of the blocks $B \in \mathcal{A}_L$. Then one of $2^3 \circ 2^2$, D_{16}, $OMLHOUSE$ is a subalgebra of L or $MO3$ is the homomorphic image of a subalgebra of L where $L_1 \circ L_2$ means the horizontal sum of two OMLs L_1 and L_2 [Figure 1].*

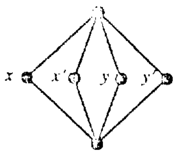
LEMMA (2.15). *If L is an irreducible, homogeneous, strictly path-connected and $n(L) \geq 2^4$, then there exists $d \in L$ such that $n([0,d]) = 2^3$ and $[0,d]$ is irreducible.*

THEOREM (2.16). *Every finite (height) OML L which does not belong to the variety $[MO2]$ generated by $MO2$ has one of the OML $MO3$, $2^3 \circ 2^2$, D_{16}, $OMLHOUSE$ as the homomorphic image of a subalgebra of L.*

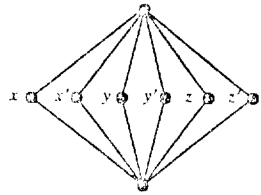
We have an extention of theorem (2.16).

THEOREM (2.17). *Every path-connected OML L with a maximal Boolean factor which does not belong to $[MO2]$ has one of the OML $MO3$, $2^3 \circ 2^2$, D_{16}, $OMLHOUSE$ as the homomorphic image of a subalgebra of L.*

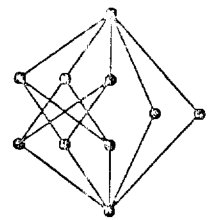
Proof. We may assume that L is irreducible by theorem (2.12) and $n(L) \geq 2^4$ by lemma (2.14).



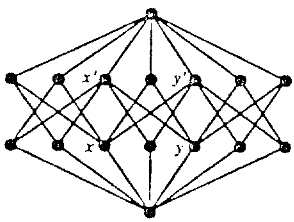
MO2



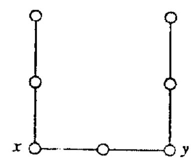
MO3



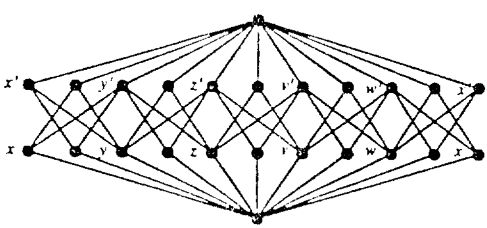
$2^3 \circ 2^2$



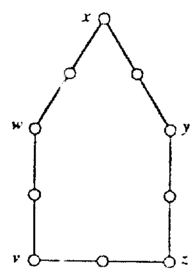
D_{16}



Greechie Diagram of D_{16}



OMLHOUSE



Greechie Diagram of OMLHOUSE

[Figure 1]

Assume first L is not strictly path-connected. Then there exist two blocks $A, B \in \mathcal{A}_L$ such that $A \cup B \leq L$ and $A \cap B = \{0,1\} = C(L)$ since L is path-connected. We have three cases: (1) $|A| = |B| = 2^2$, (2) $|A| = 2^2$ and $|B| \geq 2^3$, and (3) $|A| \geq 2^3$ and $|B| \geq 2^3$.
Case (1); There is another block C in L. Thus $MO3$ is a subalgebra of L, since A, B and C are horizontal summands.
Case (2); A is path-connected to all blocks in L. Let S be an eight-element subalgebra of B. Then $a \vee b = 1$ and $a \wedge b = 0$ hold for all $a \in A \setminus \{0,1\}$ and $b \in S \setminus \{0,1\}$ since A and B are horizontal summands. Then $A \cup S$ is a subalgebra of L isomorphic with the OML $2^3 \circ 2^2$.
Case (3); Let a is an arbitrary element of $a \in A \setminus \{0,1\}$, and let S be an eight-element subalgebra of B. Then $\{a, a'\} \cup S$ is a subalgebra of L isomorphic with the OML $2^3 \circ 2^2$.

Assume next L is not homogeneous. Then there exists two blocks C, D such that $C \cup D \leq L$, $C \cap D = S_e \cap (C \cup D)$, and $n([0,e]) \geq 2^3$. Then $2^3 \circ 2^2$ is a subalgebra of $(C \cup D)[0,e]$ since $C \cup D = (C \cup D)[0,e] \oplus (C \cup D)[0,e']$ and $C[0,e] \cap D[0,e] = \{0,e\}$. Thus $2^3 \circ 2^2$ is homomorphic image of a sugalgebra $C \cup D$ of L.

Finally, if L is homogeneous and strictly path-connected, then the conclusion follows from lemma (2.15). ■

We have the following corollary since each commutator finite OML is a path-connected OML [2 & 8] with a maximal Boolean factor [5].

COROLLARY (2.18). *Every commutator finite OML which does not belong to* $[MO2]$ *has one of* $MO3$, $2^3 \circ 2^2$, D_{16}, $OMLHOUSE$ *as the homomorphic image of a subalgebra* L.

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