ON RELATIVE CHINESE REMAINDER THEOREM

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Previously T. Porter [3] has given a relative Chinese Remainder Theorem under the hypothesis that given ring $R$ has at least one $\tau$-closed maximal ideal (by his notation $\text{Max}_\tau(R) \neq \phi$). In this short paper we drop his overall hypothesis that $\text{Max}_\tau(R) \neq \phi$ and give the proof and some related results with this Theorem.

In this paper $R$ will always denote a commutative ring with identity element and all modules will be unitary left $R$-modules unless otherwise specified.

Let $\tau$ be a given hereditary torsion theory for left $R$-module category $R$-Mod. The class of all $\tau$-torsion left $R$-modules, denoted by $\mathcal{J}$ is closed under homomorphic images, submodules, direct sums and extensions. And the class of all $\tau$-torsionfree left $R$-modules, denoted by $\mathcal{F}$, is closed under taking submodules, injective hulls, direct products, and isomorphic copies([2], Proposition 1.7 and 1.10).

Notation and terminology concerning (hereditary) torsion theories on $R$-Mod will follow [2]. In particular, if $\tau$ is a torsion theory on $R$-Mod, then a left $R$-submodule $N$ of $M$ is said to be $\tau$-closed ($\tau$-dense, resp.) submodule of $M$ if and only if $M/N$ is $\tau$-torsionfree ($\tau$-torsion, resp.). A module $M$ is called $\tau$-cocritical if $M \in \mathcal{F}$ and $M/N \in \mathcal{J}$ for each nonzero submodule $N$ of $M$. A left ideal $L$ of $R$ is $\tau$-critical if $R/L$ is $\tau$-cocritical.

Follow Porter [3], we denote $\text{Max}_\tau(M)$ be the set of all maximal $\tau$-closed submodules of $M$ and we say ideals $I, J$ are $\tau$-comaximal if $I + J$ is $\tau$-dense in $R$. Let $I_1, I_2, \ldots, I_n$ be ideals of $R$, they are pairwise $\tau$-comaximal in case $I_i + I_j$ is $\tau$-dense in $R$ whenever $i \neq j$. For example, if each $I_i$ is a maximal $\tau$-closed ideal of $R$ or each $I_i$ is a $\tau$-critical ideal, then these ideals are pairwise $\tau$-comaximal.

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The following Lemma 1 and Theorem 2 can be found in [3], we give the proof of Lemma 1 for the completeness of this paper.

**Lemma 1.** (Porter, [3]) Let $M$ be a left $R$-module, and $I, J$ be $\tau$-comaximal ideals in $R$, then $(IM \cap JM)/IJM$ is $\tau$-torsion.

**Proof.** If $x \in IM \cap JM$, $(I + J)x \in IJM$. Since $I + J$ is $\tau$-dense in $R$, we have that $ann(x + IJM)$ is $\tau$-dense in $R$. As $x$ was arbitrary we find $ann((IM \cap JM)/(IJ)M) \supseteq I + J$. Thus we have the desired result.

The author can find the following relative Chinese Remainder Theorem in [3]. The version of Porter gave us an impression to study it.

**Theorem 2 (Porter).** Let $R$ be a commutative ring and $\tau$ be a torsion theory on $R$-Mod. Suppose that $\text{Max}_\tau(R) \neq \emptyset$ and let $\{I_i|i = 1, 2, \cdots, n\}$ be a finite family of pairwise $\tau$-comaximal ideals in $R$. For any left $R$-module $M$, we have

1. $(\prod_{i=1}^n I_i)M \longrightarrow (\cap_{i=1}^n I_i)M$ is $\tau$-surjective and
2. $M \longrightarrow \oplus_{i=1}^n M/I_iM$ is $\tau$-surjective with kernel $\cap_{i=1}^n I_iM$

The condition $\text{Max}_\tau(R) \neq \emptyset$ was used by the fact that every member in $\text{Max}_\tau(R)$ is prime ideal in $R$, which is Albu and Năstăsescu’s work [1].

In order to drop the condition $\text{Max}_\tau(R) \neq \emptyset$, we need a lemma, which is useful in the proof of main Theorem.

**Lemma 3.** Let $R$ be a commutative ring and $\{I_i|i = 1, 2, \cdots, n\}$ be pairwise $\tau$-comaximal ideals of $R$. Let $M$ be any left $R$-module, then we have the following:

1. $I_i \cap \text{j}_{i \neq i} I_j$ is $\tau$-dense in $R$ for each $i = 1, 2, \cdots, n$.
2. $I_iM + (\cap_{j \neq i} I_j)M$ is $\tau$-dense in $M$ for each $i = 1, 2, \cdots, n$.

**Proof.** (1) We prove for the case $I_1 + D_1$ is $\tau$-dense in $R$, where $D_1 = \cap_{j \neq 1} I_j$. For the case $n = 1$ is clear. Assume that $I_1 + \cap_{j=2}^k I_j$ is $\tau$-dense in $R$.

Note that $I_1 + \cap_{j=2}^{k+1} I_j$ contains $(I_1 + \cap_{j=2}^k I_j)(I_1 + I_{k+1})$, which is $\tau$-dense in $R$, thus $I_1 + \cap_{j=2}^{k+1}$ is $\tau$-dense in $R$ i.e., the induction step is proved. Consequently $I_1 + \cap_{j=2}^n I_j = I_1 + D_1$ is $\tau$-dense in $R$. A similar
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argument shows that for each \( i = 1, 2, \cdots, n \), \( I_i + D_i \) is \( \tau \)-dense in \( R \), where \( D_i = I_1 \cap I_2 \cap \cdots \cap I_{i-1} \cap I_{i+1} \cap \cdots \cap I_n \).

(2) For each \( i = 1, 2, \cdots, n \), note that \( I_i M + D_i M = (I_i + D_i)M \). 
\( M/(I_i + D_i)M \) can be a left \( R/(I_i + D_i) \)-module by the action 
\( (\tau + I_i + D_i)(m + (I_i + D_i)M) = \tau m + (I_i + D_i)M \)

We regard \( M/(I_i + D_i)M \) as a homomorphic image of free \( R/(I_i + D_i) \)-module \( \bigoplus_{\alpha \in M} (R/(I_i + D_i))_{\alpha} \), by (1) \( R/(I_i + D_i) \) is \( \tau \)-torsion and \( \tau \)-torsion class is closed under direct sum, we have that \( I_i M + D_i M \) is \( \tau \)-dense in \( M \).

**Theorem 4. (Relative Chinese Remainder Theorem).** Let \( R \) be a commutative ring and \( \{I_i | i = 1, 2, \cdots, n \} \) be a finite family of pairwise \( \tau \)-comaximal ideals in \( R \). For any left \( R \)-module \( M \), we have

1. \( \prod_{i=1}^{n} I_i M \rightarrow (\cap_{i=1}^{n} I_i) M \) is \( \tau \)-surjective and
2. \( M \rightarrow \bigoplus_{i=1}^{n} M/I_i M \) is \( \tau \)-surjective with kernel \( \cap_{i=1}^{n} I_i M \)

**Proof.** (1) The case \( n = 1 \) is trivial. Assume the result holds for any left \( R \)-module \( M \) and all families of pairwise \( \tau \)-comaximal ideals having fewer than \( n \). Consider \( \{I_i | i = 1, 2, \cdots, n \} \) and we denote by 
\( P_i = \prod_{j \neq i} I_j \) and \( D_i = \cap_{j \neq i} I_j \) We want to show that \( I_i + P_i \) is \( \tau \)-dense in \( R \). By Lemma 3 (1), for each \( i = 1, 2, \cdots, n \), \( I_i \) and \( D_i \) is \( \tau \)-comaximal ideals in \( R \). Now apply to Lemma 1, we have that \( \frac{I_i + D_i}{I_i + D_i} \) is \( \tau \)-torsion, so its homomorphic image \( \frac{I_i + D_i}{I_i + P_i} \) is \( \tau \)-torsion. Consider the following short exact sequence,

\[
0 \rightarrow \frac{I_i + D_i}{I_i + P_i} \rightarrow \frac{R}{I_i + P_i} \rightarrow \frac{R}{I_i + D_i} \rightarrow 0
\]

By the Lemma 3(1), \( R/(I_i + D_i) \) is \( \tau \)-torsion module. And the \( \tau \)-torsion class is closed under extension, so we have \( R/(I_i + P_i) \) is \( \tau \)-torsion, thus \( I_i + P_i \) is \( \tau \)-dense in \( R \).

Now we can apply the Lemma 1, and get

\[
\left( \prod_{k=1}^{n} I_k \right) M = I_i P_i M \rightarrow I_i M \cap P_i M \text{ is an } \tau \text{-epimorphism.}
\]

Now by the induction hypothesis, \( I_i M \cap P_i M \rightarrow I_i M \cap (D_i M) \) is \( \tau \)-surjection.

Thus \( \left( \prod_{k=1}^{n} I_k \right) M \rightarrow (\cap_{k=1}^{n} I_k) M \) is \( \tau \)-surjection.

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(2) The case \( n = 1 \) is clear. We also assume the result holds for any left \( R \)-module \( M \) and all families of pairwise \( \tau \)-comaximal ideals having fewer than \( n \).

Consider the following short exact sequence

\[
0 \longrightarrow \frac{M}{(\bigcap_{j \neq i} I_j M) \cap I_i M} \longrightarrow \frac{M}{D_i M} \oplus \frac{M}{I_i M} \longrightarrow \frac{M}{D_i M + I_i M} \longrightarrow 0
\]

By the Lemma 3(2), \( M/(D_i M + I_i M) \) is \( \tau \)-torsion. Thus \( M/(D_i M \cap I_i M) \) is \( \tau \)-dense in \( M/D_i M \oplus M/I_i M \). Now apply the induction hypothesis

\[
\frac{M}{D_i M} \oplus \frac{M}{I_i M} \longrightarrow \bigoplus_{j \neq i} \frac{M}{I_j M} \oplus \frac{M}{I_i M} \cong \bigoplus_{i=1}^n \frac{M}{I_i M}
\]

is \( \tau \)-surjection. Thus we have the desired result.

We examine \( R \)-submodules \( \{I_i M| i = 1, 2, \cdots, n\} \) of \( M \) in above lemmas and theorems, and consider the following concept in module theoretic sense.

**Definition.** Let \( M \) be a left \( R \)-module, a set of left \( R \)-submodules of \( M \) \( \{N_i|i = 1, 2, \cdots, n\} \) is called \( \tau \)-co-indepenent in \( M \) if (i) each \( N_i \) is not \( \tau \)-dense in \( M \) and (ii) \( N_i + \bigcap_{j \neq i} N_j \) is \( \tau \)-dense in \( M \) for each \( i = 1, 2, \cdots, n \).

For example, given pairwise \( \tau \)-commaximal ideals of commutative ring \( R \) \( \{I_i|i = 1, 2, \cdots, n\} \), consider left \( R \)-submodules \( \{I_i M| i = 1, 2, \cdots, n\} \), then the Lemma 3(2) shows that \( \{I_i M| i = 1, 2, \cdots, n\} \) is a set of \( \tau \)-co-indepenent in \( M \).

Properties on \( \tau \)-co-indepenent submodules can be found in [4].

In here, we mention only the fact related with Relative Chinese Remainder Theorem.

**Proposition 5.** Let \( R \) be a ring with identity (\( R \) may not be commutative) and let \( \{N_i|i = 1, 2, \cdots, n\} \) be a set of \( \tau \)-co-indepenent \( R \)-submodules of \( M \). Then we have \( M \longrightarrow \bigoplus_{i=1}^n \frac{M}{N_i} \) is \( \tau \)-surjectivewith kernel \( \bigcap_{i=1}^n N_i \).
Proof. The case $n = 1$ is clear. We assume for any left $R$-module $M$ and all families of $\tau$-coindependent submodules having less than $n$. Consider the following short exact sequence:

$$
0 \longrightarrow \frac{M}{(\cap_{i=1}^{n-1} N_i) \cap N_n} \longrightarrow \frac{M}{\cap_{i=1}^{n-1} N_i} \oplus \frac{M}{N_n} \longrightarrow \frac{M}{\cap_{i=1}^{n-1} N_i + N_n} \longrightarrow 0
$$

By the $\tau$-coindependency of $\{N_i : i = 1, 2, \cdots, n\}$, $\cap_{i=1}^{n-1} N_i + N_n$ is $\tau$-dense in $M$. Use the induction hypothesis we have the result.

Corollary 6. If $\text{Max}_\tau(M)$ is finite, then $M/J_\tau(M)$ is $\tau$-semisimple $\tau$-artinian, where $J_\tau(M)$ is the relative Jacobson radical of $M$.

Proof. Since $\text{Max}_\tau(M)$ is finite, $J_\tau(M) = \cap_{i=1}^{n} N_i$, where $N_i$ is $\tau$-critical submodules of $M$. And the set $\{N_i : i = 1, 2, \cdots, n\}$ forms a $\tau$-coindependent submodules in $M$, then the relative Chinese Remainder Theorem (Theorem 4) gives an $\tau$-epimorphism $\frac{M}{J_\tau(M)} \xrightarrow{\oplus} \frac{M}{\oplus_{i=1}^{n} N_i}$.

Hence $\frac{M}{J_\tau(M)}$ is $\tau$-semisimple and $\tau$-artinian as left $R$-module.

References


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