THE GROUP OF UNITS IN A LEFT ARTINIAN RING

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Let $R$ be a left Artinian ring with identity 1 and let $G$ be the group of units of $R$. It is shown that if $G$ is finite, then $R$ is finite. It is also shown that if 2.1 is a unit in $R$, then $G$ is abelian if and only if $R$ is commutative.

1. Introduction and basic definitions

An element $a$ in $R$ is said to be left quasi-regular if there exists $r \in R$ such that $r + a + ra = 0$. In this case, the element $r$ is called a left quasi-inverse of $a$. A (right, left or two-sided) ideal $I$ of $R$ is said to be left quasi-regular if every element of $I$ is left quasi-regular. Similarly, $a \in R$ is said to be right quasi-regular if there exists $r \in R$ such that $a + r + ar = 0$. Right quasi-inverse and right quasi-regular ideals are defined analogously. It is clear that if $R$ has an identity 1, then $a$ is left [resp. right] quasi-regular if and only if $1 + a$ is left [resp. right] invertible. The Jacobson radical $J$ of $R$ is defined by the left quasi-regular left ideal which contains every left quasi-regular left ideal of $R$. A ring $R$ is said to be semisimple if its Jacobson radical $J$ is zero. We note that $R/J$ is semisimple.

In [2], Wedderburn-Artin have shown that if $R$ is a semisimple left Artinian ring, then $R$ is isomorphic to a direct sum of a finite number of simple rings. Hence we obtain the following:

THEOREM 1.1. If $R$ is a left Artinian ring with identity, then $R/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring $D_i$ for each $i = 1, 2, \ldots, n$ and for some a positive integer $n$.

Proof. See [2, Theorem 2.14, p.431 and Theorem 3.3, p.435].

2. Properties of $R$ when $G$ is finite and abelian

In this section, we shall denote $G$ by the group of units of $R$ and denote $J$ by the Jacobson radical of $R$.

We begin with the following lemma:

**Lemma 2.1.** Let $R$ be a ring, and let $G^*$ be the group of units of $R/J$. Then $g \in G$ if and only if $g + J \in G^*$.

**Proof.** ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that $g^* = g + J \in G^*$. Then there exists $h^* = h + J \in G^*$ such that $g^*h^* = h^*g^* = 1^*$ where $1^*$ is the identity of $G^*$. So $1 - hg \in J$. By the definition of $J$, $1 + J \subseteq G$ and so $gh$ and $hg \in G$. It is clear that $g \in G$.

**Lemma 2.2.** Let $R$ be a ring with identity. Then $a \in R$ is left quasi-regular if and only if $a + J \in R/J$ is left quasi-regular.

**Proof.** It follows easily from Lemma 2.1.

**Theorem 2.3.** Let $R$ be a left Artinian ring with identity $1$. If $G$ is finite group, then $R$ is finite.

**Proof.** By Theorem 1.1, $R/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring $D_i$ for each $i = 1, 2, \ldots, n$ and for some a positive integer $n$. If $G$ is finite, then by Lemma 2.1, $G^*$, the group of units of $R/J$, is also finite. Then $D_i$ is finite for each $i = 1, 2, \ldots, n$. Indeed, suppose that $D_i$ is infinite for some $i$. For simplicity of notation, we can assume $R/J = \bigoplus_{i=1}^{n} M_i(D_i)$. Consider a subset $G_i^* = \bigoplus_{i=1}^{n} H_i$ where $H_j = \{e_j\}, (e_j$ is the identity of $M_j(D_j)$ for $j \neq i$ and $H_i = \{(a_{st}) \in M_i(D_i) : a_{11} \in D_i \setminus \{0_i\}, a_{ss} = e_i, (2 \leq s \leq n_i), a_{st} = 0_i (2 \leq s, t \leq n_i, s \neq t)$ and $e_i$ (resp. $e_i$) is zero (resp. identity) of $D_i\}$. Then $G_i^*$ is a subgroup of $G^*$ and $|G_i^*| = |D_i \setminus \{0_i\}|$ is infinite, which contradicts to the fact that $G^*$ is finite group. Hence $D_i$ is finite for each $i = 1, 2, \ldots, n$, and so $R/J$ is finite. Since $1 + J \subseteq G$ and $G$ is finite, $J$ is finite. Hence $|R| = |J| \cdot |R/J|$ is finite.

**Lemma 2.4.** Let $R$ be a ring with identity and let $G$ be the group of units of $R$. If $G$ is abelian group and $a$ and $b$ are quasi-regular elements of $R$, then $ab = ba$. In particular, $J$ is commutative.
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Proof. Since $1 + J \subseteq G$ and $a$ and $b \in J$, $(1 + a)(1 + b) = (1 + b)(1 + a)$. Hence $ab = ba$. Since each element of $J$ is quasi-regular, $J$ is commutative.

Remark. In Theorem 2.3, the condition that $R$ has identity is necessary because $p$-Prüfer ring $Z(p^\infty)$ is infinite Artinian ring without identity which has no units.

Lemma 2.5. Let $R$ be a left Artinian ring with identity. If $G$ is abelian group, then $R/J \cong \bigoplus_{i=1}^{n} F_i$ where $F_i$ is a field for each $i = 1, 2, \ldots, n$ and for some positive integer $n$.

Proof. By Theorem 1.1, $R/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring $D_i$ for each $i = 1, 2, \ldots, n$ and for some a positive integer $n$. First, we will show that each $D_i$ is a field. Consider the subgroup $G_i^* = \bigoplus_{i=1}^{n} H_i$ of $G^*$ given in the proof of Lemma 2.4. Since $G^*$ is abelian, $H_i$ is also abelian, and so $D_i$ is abelian, that is, $D_i$ is field. Let $D_i = F_i$. Next, we will show that $n_i = 1$ for each $i$. Assume that $n_i \geq 2$ for some $i$. Consider two elements $a = (a_{st})$ and $b = (b_{st})$ in $M_i(F_i)$ where if $s = t, a_{12} = a_{st} = c_i$, otherwise $a_{st} = 0_i$, and if $s = t, b_{21} = b_{st} = 1_i$, otherwise $b_{st} = 0_i$. By the simple calculation, we have $(1, 1)$-entry of $ab = 2 \neq 1 = (1, 1)$-entry of $ba$. Thus the group of units in $M_i(F_i)$ is not abelian, and so $G^*$ is not abelian group, which is a contradiction. Hence we have the result.

Let $R$ be a left Artinian ring with identity such that $G$ is abelian group. By Lemma 2.5, $R/J \cong \bigoplus_{i=1}^{n} F_i$ where $F_i$ is field for each $i$ ($1 \leq i \leq n$) and for some positive integer $n$. For simplicity of notation, we can assume that $R/J \cong \bigoplus_{i=1}^{n} F_i$. Let $\phi : R \rightarrow R/J$ denote the canonical epimorphism and for each $i$, let $R_i = \phi^{-1}(\bigoplus_{j=1}^{n} H_j)$ where $H_j = \{0\} \{0_j\}$ $(0_j$ is additive identity of $F_j)$ for $j \neq i$ and $H_i = F_i$. Let $\phi_i = \phi|_{R_i}$. Then Ker $\phi_i = \{a \in R_i : \Pi_i(\phi_i(a)) = 0_i\}$ where $\Pi_i$ is the projection of $F_j$ to $F_i$. Note that Ker $\phi_i = J$ for each $i = 1, 2, \ldots, n$ and each $R_i$ is an ideal of $R$. If $1_i$ is the identity of $F_i$, let $1_i^*$ denote the identity of $\phi_i = \bigoplus_{i=1}^{n} H_j$, that is, $1_i^* = \bigoplus_{i=1}^{n} a_j$ where $a_j = \phi_j$ for $j \neq i$ and $a_i = 1_i$. Observe that $\phi_i^{-1}(\{1_i^*\})$ is contained in the center of $R_i$ if and only if $\phi_i^{-1}(\{-1_i^*\})$ is contained in the center of $R_i$.

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Lemma 2.6. Let \( \phi : R \to R' \) be a ring epimorphism. If \( A \) and \( B \) are subsets of \( R' \), then \( \phi^{-1}(A + B) = \phi^{-1}(A) + \phi^{-1}(B) \).

Proof. If \( x \in \phi^{-1}(A + B) \), then \( \phi(x) = a + b \in A + B \). Since \( \phi \) is onto, there exist \( a* \in A \) and \( b* \in B \) such that \( \phi(a*) = a \) and \( \phi(b*) = b \). So \( \phi(x) = a + b = \phi(a*) + \phi(b*) = \phi(a* + b*) \in \phi(\phi^{-1}(A) + \phi^{-1}(B)) \). Hence \( x \in \phi^{-1}(A) + \phi^{-1}(B) \).

If \( x \in \phi^{-1}(A) + \phi^{-1}(B) \), then \( x = a* + b* \) where \( a* \in \phi^{-1}(A) \) and \( b* \in \phi^{-1}(B) \). So \( \phi(x) = \phi(a* + b*) = \phi(a*) + \phi(b*) \in A + B \). Hence \( x \in \phi^{-1}(A + B) \).

Lemma 2.7. If \( R \) is a left Artinian ring with identity, then \( R = R_1 + R_2 + \cdots + R_n \) where \( R_i = \phi^{-1}(\oplus_{j=1}^n H_i) \) with \( H_j = \{0\} \) (0 is additive identity of \( F_j \)) for \( j \neq i \) and \( H_i = F_i \).

Proof. Let \( F_i^* = \oplus_{j=1}^n H_i \) for each \( i \). Then \( \oplus_{i=1}^n F_i = F_1^* + F_2^* + \cdots + F_n^* \). Hence \( R = \phi^{-1}(R) = \phi^{-1}(R/J) = \phi^{-1}(\oplus F_i) = \phi^{-1}(F_1^* + F_2^* + \cdots + F_n^*) = \phi^{-1}(F_1^*) + \phi^{-1}(F_2^*) + \cdots + \phi^{-1}(F_n^*) = R_1 + R_2 + \cdots + R_n \) by Lemma 2.6.

Lemma 2.8. Let \( R \) be a ring with identity such that \( G \) is abelian group and \( R/J = \oplus_{i=1}^n F_i \) where each \( F_i \) is field. If \( \phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i) \) (center of \( R_i \)), then \( R \) is commutative.

Proof. Since \( R_i \) is an ideal of \( R \), if \( a \in R_i \), then \( a \) is quasi-regular in \( R_i \) if and only if \( a \) is quasi-regular in \( R \). Hence by Lemma 2.2, if \( a \in R_i \), then \( a \) is quasi-regular in \( R_i \) if and only if \( \phi(a) \) is quasi-regular in \( R/J \), that is, \( \phi_i(a) \) is quasi-regular in \( F_i^* = \oplus_{j=1}^n H_j \) where \( H_j = \{0\} \) for \( j \neq i \) and \( H_i = F_i \). Hence for \( a \in R_i \), \( a \) is quasi-regular if and only if \( \Pi_i(\phi_i(a)) + 1_i \neq 0_i \).

Now let \( a, b \in R_i \). If \( a \) and \( b \) are quasi-regular, then \( ab = ba \) by Lemma 2.4. If \( a \) is not quasi-regular, then \( \Pi_i(\phi_i(a)) + 1_i = 0_i \), that is, \( a \in \phi_i^{-1}(\{-1_i^*\}) \). Thus \( a \) is in the center of \( R_i \) and so \( ab = ba \). Similarly, if \( b \) is not quasi-regular, then \( ab = ba \).

Lemma 2.9. Let \( R \) be a ring with identity such that \( G \) is abelian group and \( R/J = \oplus_{i=1}^n F_i \) where each \( F_i \) is field. If \( \phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i) \) (center of \( R_i \)) for all \( i = 1, 2, \cdots, n \), then \( R \) is commutative.

Proof. Let \( a \in R_i \) and \( b \in R_j \) for \( i \neq j \) (1 \( \leq i, j \leq n \)). By Lemma 2.7, it suffices to show that \( ab = ba \). By Lemma 2.4, we may assume
that both \( a \) and \( b \) are not quasi-regular. Without loss of generality, we may assume that \( a \) is not quasi-regular. Then \( \Pi_i(\phi_i(a)) = -1_i \). Since \( ab = ba \) if and only if \((-a)b = b(-a)\), we may assume that \( \Pi_i(\phi_i(a)) = 1_i \). Now \( ab, ba \in R_i \cap R_j \) since \( R_i \) and \( R_j \) are ideals of \( R \). But for \( i \neq j \), \( R_i \cap R_j = J \). So \( ab, ba \in J \). Since \( J \subseteq Z(R_i) \) for each \( i \), \( ab \) and \( ba \) are in \( Z(R_i) \) for each \( i \). Hence \( a(ab) = (ab)a = a(ba) = (ba)a \), that is \( a^2b = ba^2 \). Since \( \Pi_i(\phi_i(a^2 - a)) = 0_i \), \( a^2 - a \in J \). So \( (a^2 - a) = b(a^2 - a) \). Hence \( -ab = -ba \), that is, \( ab = ba \).

**Lemma 2.10.** Let \( R \) be a ring with identity such that \( G \) is abelian group and \( R/J = \bigoplus_{i=1}^n F_i \) where each \( F_i \) is field. If \( \text{char} \ (F_i) \neq 2 \) for some \( i \), then \( \phi_i^{-1}(\{1_i^2\}) \subseteq Z(R_i) \) (= center of \( R_i \)).

**Proof.** Since \( \text{char} \ (F_i) \neq 2 \) for some \( i \), \( 1_i \neq -1_i \). For any \( u_i \in F_i \setminus \{0_i, -1_i\} \), there exists \( w/i \in F_i \setminus \{0_i, -1_i\} \) such that \( u_i \cdot w_i = 1_i \). So \( w_i + 1_i \neq 0_i \) and \( u_i + 1_i \neq 0_i \), and hence \( u_i \) and \( w_i \) are quasi-regular elements of \( F_i \). Let \( u = (0_i, \ldots, 0_{i-1}, u_i, 0_{i+1}, \ldots, 0_n) \) and \( w = (0 - 1_i, \ldots, 0_{i-1}, w_i, 0_{i+1}, \ldots, 0_n) \). Then \( u \) and \( w \) are quasi-regular in \( \bigoplus H_j \) where \( H_j = \{0_j\} \) for \( j \neq i \) and \( H_i = F_i \). Since \( \phi_i \) is onto, there exist \( a, b \) and \( e \in R_i \) such that \( \phi_i(a) = u \), \( \phi_i(b) = w \) and \( \phi_i(e) = 1_i \). Then \( \Pi_i(\phi_i(e - ab)) = \Pi_i(1_i - uw) = 0_i \), so \( e - ab \in \text{Ker} \phi_i = J \). Note that \( a \) and \( b \) are quasi-regular in \( R \) if and only if \( \phi(a) \) and \( \phi(b) \) are quasi-regular in \( R/J \). Let \( x \) be arbitrary element of \( R_i \). If \( x \) is quasi-regular, then by Lemma 2.4, \( x(e - ab) = (e - ab)x \) since \( e - ab \in J \). Hence \( xe - xab = ex - abx \). Since \( a \) and \( b \) are quasi-regular, \( xab = abx \). Thus \( xe = ex \). If \( x \) is not quasi-regular, then \( \Pi_i(\phi_i(x)) = -1_i = \Pi_i(\phi_i(-e)) \). So \( x + e \in \text{Ker} \phi_i = J \). Thus \( x + e = j \) for some \( j \in J \). Since \( j \) is quasi-regular in \( R_i \), \( ej = je \). So \( xe = (j - e)e = jes = ej - e^2 = e(j - e) = ex \). Thus \( \phi_i^{-1}(\{1_i^2\}) \subseteq Z(R_i) \).

**Theorem 2.11.** Let \( R \) be a left Artinian ring with identity 1 such that \( 2 = 2 \cdot 1 \) is a unit in \( R \). Then \( G \) is abelian if and only if \( R \) is commutative.

**Proof.** \(( \Leftarrow \) Clear.

\(( \Rightarrow \) Suppose that \( G \) is abelian. Then \( R/J \cong \bigoplus_{i=1}^n F_i \), where \( F_i \) is a field for each \( i = 1, 2, \ldots, n \) and for some positive integer \( n \). For simplicity of notation, we can assume that \( R/J \cong \bigoplus_{i=1}^n F_i \). Since 2 is
unit in $R$, then $2 + J$ is unit in $R/J$ by Lemma 2.1. So char($F_i$) $\neq 2$ for each $i = 1, 2, \ldots, n$. Therefore, the theorem follows from Lemma 2.6, Lemma 2.9 and Lemma 2.10.

**Remark.** In Theorem 2.11, the condition that 2 is a unit in $R$ is essential, since the ring $R$ of upper triangular $2 \times 2$ matrices over $Z_2$ is not commutative but the group of units of $R$ is abelian.

**Corollary 2.12.** Let $R$ be a left Artinian ring with identity 1 such that $2 = 2 \cdot 1$ is a unit in $R$. If $G$ is cyclic, then $R$ is a finite commutative ring.

**Proof.** If $G$ is cyclic, $G$ is abelian. So by Theorem 2.11 $R$ is commutative. Moreover, if $G$ is cyclic, then $R$ is finite. [See [3]]

**References**


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