W–REGULAR CONVERGENCE OF $R^i$–CONTINUA

C.J. Rhee*, I.S. Kim and R.S. Kim

1. Introduction and basic definitions

In the course of study of dendroids, Czuba [3] introduced a notion of $R^i$-continua which is a generalization of $R$-arc [1]. He showed a new class of non-contractible dendroids, namely of dendroids which contain an $R^i$-continuum. Subsequently Charatonik [2] attempted to extend the notion into hyperspace $C(X)$ of metric continuum $X$. In so doing, there were some oversights in extending some of the results relating $R^i$-continua of dendroids for metric continua. In fact, Proposition 1 in [2] is false (see example C below) and his proof of Theorem 6 in [2] is not correct (Take Example 4 in [4] with $K = [e, e']$ as an $R^i$-continuum of $X$ and work it out. Then one sees that $K$ $\notin K$ as he claimed otherwise.).

The aims of this paper are to introduce a notion of $w$-regular convergence which is weaker than 0-regular convergence and to prove that the $w$-regular convergence of a sequence $\{X_n\}_{n=1}^\infty$ to $X_0$ of subcontinua of a metric continuum $X$ is a necessary and sufficient for the sequence $\{C(X_n)\}_{n=1}^\infty$ to converge to $C(X_0)$, and also to prove that if a metric continuum $X$ contains an $R^i$-continuum with $w$-regular convergence, then the hyperspace $C(X)$ of $X$ contains $R^i$-continuum.

Let $(X, d)$ be a compact metric space. Let $2^X = \{ A \subset X : A$ is nonempty and closed $\}$ and let $C(X) = \{ A \in 2^X : A$ is connected $\}$. For each $A \in 2^X$ and $\varepsilon > 0$, let $N(\varepsilon, A) = \{ x \in X : d(x, a) < \varepsilon$ for some $a \in A \}$. If $A, B \in 2^X$, let $H(A, B) = \inf \{ \varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A) \}$; we call $H$ the Hausdorff metric for $2^X$ (or $C(X)$) induced by $d$. The spaces $2^X$ and $C(X)$, with the Hausdorff metric, are called hyperspaces of $X$.

Received November 25, 1992. Revised April 19, 1993.
* This paper was partially supported by Korean Science and Engineering Foundation during his visit to Won Kwang University
** We thank referee for valuable suggestion.
We adopt the following notations: if \( x \in M \subseteq X \), let \( C(M) = \{ A \in C(X) : A \subseteq M \} \), \( T(x, M) = \{ A \in C(M) : x \in A \} \), and let \( M^* = \{ \{ x \} : x \in M \} \).

Let \( \{ A_n \}_{n=1}^{\infty} \) be a sequence of subsets of a space \( X \). Let \( LiA_n \) be the set of all \( x \in X \) such that if \( U \) is a neighborhood of \( x \), then \( U \cap A_n \neq \emptyset \) for infinitely many \( n \). If \( LiA_n = LsA_n = A \), then we say that the sequence \( \{ A_n \}_{n=1}^{\infty} \) converges to \( A \), written \( LtA_n = A \) or \( A_n \to A \). It is known [7] that if \( \{ A_n \}_{n=1}^{\infty} \) is a sequence in the hyperspace \( 2^X \) (or \( C(X) \)) of the metric continuum \( X \), then \( A_n \to A \) if and only if \( H(A_n, A) \to 0 \) as \( n \to \infty \). For further properties of these definitions, we refer to [9].

2. Convergence of \( \{ C(X_n) \}_{n=1}^{\infty} \)

Let \( \{ X_n \}_{n=1}^{\infty} \) be a sequence of subcontinua of a metric continuum \( X \) which converges to \( X_0 \). One may ask under what condition imposed on the sequence so that \( \{ C(X_n) \}_{n=1}^{\infty} \) converges to \( C(X_0) \). In [9], 0-regular convergence was given. However, this condition is sufficient but not necessary. We provided here one simple condition, call it \( w \)-regular convergence, which is both necessary and sufficient.

**Definition 2.1** [9]. A sequence \( \{ X_n \}_{n=1}^{\infty} \) of subsets of a metric continuum \( X \) is said to converge 0-regularly to \( X_0 \) provided that the following two conditions are satisfied:

(a) \( X_n \to X_0 \) as \( n \to \infty \);

(b) Given \( \varepsilon > 0 \), there exist \( \delta(\varepsilon) > 0 \) and a positive integer \( N \) such that if \( n \geq N \), then any two points of \( X_n \) less than \( \delta \) apart lie together in a connected subset of \( X_n \) of diameter less than \( \varepsilon \).

**Theorem 2.2** [6]. If \( \{ X_n \}_{n=1}^{\infty} \) is a sequence of subcontinua of a metric continuum \( X \) such that \( X_n \to X_0 \) 0-regularly, then \( \{ C(X_n) \}_{n=1}^{\infty} \) converges to \( C(X_0) \) with respect to the Hausdorff metric.

**Definition 2.3.** A sequence \( \{ X_n \}_{n=1}^{\infty} \) of subsets of a metric continuum \( X \) is said to converge \( h \)-regularly to \( X_0 \) if it satisfies the following two conditions:

(a) \( X_n \to X_0 \);
W-regular convergence of $R^1$-continua

(b) Given $\varepsilon > 0$ and $A \in T(x, X_0)$, there exist $\delta > 0$ and a positive integer $N$ such that each point $y \in X_n \cap V(x)$, where $V(x)$ is the $\delta$-neighborhood of $x$, has an element $B \in T(y, X_n)$ such that $H(A, B) < \varepsilon$ for each $x \in X$ and for $n \geq N$.

**Lemma 2.4** [6,9]. Let $A$ be a metric continuum. For each $\varepsilon > 0$, there is a finite set $F = \{a_1, a_2, \ldots, a_n\} \subset A$ such that

1. $H(F, A) < \varepsilon$, and
2. the distance between any two consecutively indexed points of $F$ is less than $\varepsilon$.

**Lemma 2.5.** If the sequence $\{X_n\}_{n=1}^\infty$ of subcontinua of a metric continuum $X$ converges 0-regularly, then it converges $h$-regularly.

**Proof.** Let $\{X_n\}_{n=1}^\infty$ be a sequence of subcontinua of a metric space $X$ which converges to $X_0$ 0-regularly. Let $A \in T(x, X_0)$ and $\varepsilon > 0$. Since $X_n \to X_0$ 0-regularly, there exist $\delta, 0 < \delta < \varepsilon$, and a positive integer $N_1$ such that, for each $n > N_1$, if $p, q \in X_n$ such that $d(p, q) < \delta$, then there is a subcontinuum of $X_n$ containing $p$ and $q$, denoted by $B_{n}(p, q)$, such that the diameter of $B_{n}(p, q)$ is less than $\frac{\varepsilon}{2}$.

Since $X_n \to X_0$, there is a positive integer $N_2$ such that if $n > N_2$, then $H(X_n, X_0) < \frac{\delta}{3}$.

Since $A$ is a subcontinuum, we let $F = \{a_1, a_2, \ldots, a_n\} \subset A$ such that $H(F, A) < \frac{\delta}{6}$ and $d(a_s, a_{s+1}) < \frac{\delta}{6}$ for each $s = 1, 2, \ldots, t - 1$, by Lemma 2.4. Let $F' = F \cup \{x\}$. Then $H(F', A) < \frac{\delta}{6}$, and $d(x, a_i) < \frac{\delta}{6}$ for some $a_i \in F$. Let $N = \max\{N_1, N_2\}$ and let $V$ be the $\frac{\delta}{6}$-neighborhood of $x$ in $X$ and $y \in V \cap X_{n_0}$ for some $n_0 > N$.

For each $a_s \in F$, choose $y_s \in X_{n_0}$ such that $d(a_s, y_s) < \frac{\delta}{3}$. Then $d(y_s, y_{s+1}) < d(y_s, a_s) + d(a_s, a_{s+1}) + d(a_{s+1}, y_{s+1}) < \delta$. And $d(y, y_i) < d(y, x) + d(x, a_i) + d(a_i, y_i) < \delta$. Then we have subcontinua $B_{n_0}(y, y_i), B_{n_0}(y_s, y_{s+1})$ of diameter less than $\frac{\varepsilon}{2}$. Let $B_{n_0} = B_{n_0}(y, y_i) \cup_{s=1}^{t-1} B_{n_0}(y_s, y_{s+1})$. Then $B_{n_0} \in T(y, X_{n_0})$. And one can easily verify that $H(B_{n_0}, A) < \varepsilon$.

**Definition 2.6.** The sequence $\{X_n\}_{n=1}^\infty$ of subsets of a metric continuum $X$ is said to converge to $X_0$ $w$-regularly if it satisfies the follow-
ings:

1. $X_n \to X_0$, and
2. Given $\varepsilon > 0, x \in X_0$, and $A \in T(x, X_0)$, there are a $\delta > 0$ and a positive integer $N$ such that, whenever the $\delta$-neighborhood $V$ of $x \in X$ intersects $X_n, n \geq N$, then there is a point $y \in V \cap X_n$ having an element $B \in T(y, X_n)$ with $H(A, B) < \varepsilon$.

**Theorem 2.7.** $h$-regular convergence of a sequence of subcontinua of a metric continuum implies $w$-regular convergence.

**Remark 2.8.**

1. $h$-regular convergence does not imply $0$-regular convergence.
2. $w$-regular convergence does not imply $h$-regular convergence.
3. $C(X_n) \to C(X_0)$ does not imply $h$-regular convergence of $X_n \to X_0$.

We illustrate the remark by the following two examples:

**Example A.** Let $X = [0,1] \times [0,1]$. Let $p_0 = (0,0)$ and $q_0 = (1,0)$ and let $X_0 = p_0q_0$ denote the line segment between $p_0$ and $q_0$. For each positive integer $n$, let $p_n = (0, \frac{1}{n})$ and $q_n = (1, \frac{1}{n})$. For each even positive integer $n = 2m$, let $X_m = p_{n-1}q_{n-1} \cup q_{n-1}p_n \cup p_nq_n$ if $m$ is odd, and let $X_m = p_{n-1}q_{n-1} \cup p_{n-1}p_n \cup p_nq_n$ if $m$ is even. Then $X_m \to X_0$ $h$-regularly but not $0$-regularly at either $p_0$ or $q_0$.

**Example B.** We give a sequence of figures $Z$ which converges to an arc. Then the convergence of the sequence is $h$-regular but not $0$-regular. Let $p_0 = (0,0)$ and $s_0 = (1,0)$, and let $X_0 = p_0s_0$. For each positive integer $n$, let $p_n = (0, \frac{1}{2n-1}), q_n = (\frac{3}{4}, \frac{1}{2n-1}), r_n = (\frac{1}{4}, \frac{1}{2n}),$ and $s_n = (1, \frac{1}{2n})$. Let $X_n = p_nq_n \cup q_nr_n \cup r_ns_n$ for each $n$. Let $x = (\frac{1}{4}, 0)$ and $A = p_0x$. Then the convergence $X_n \to X_0$ is $w$-regular but not $h$-regular at $x$. Also it is easily seen that $LtC(X_n) = C(X_0)$.

**Theorem 2.9.** If the sequence $\{X_n\}_{n=1}^{\infty}$ of subsets of a metric continuum $X$ converges to $X_0$ $w$-regularly, then the sequence $\{C(X_n)\}_{n=1}^{\infty}$ converges to $C(X_0)$.

**Proof.** Since $LtX_n = X_0, X_0^* \subset LtC(X_n) \subset LsC(X_n) \subset C(X_0)$. Let $A \in C(X_0), a \in A$, and let $\varepsilon > 0$. Since $X_n \to X_0$ is $w$-regular,
W-regular convergence of $R^i$-continua

there exist $\delta$-neighborhood $V$ of $a$ and a positive integer $N$ such that $V \cap X_n \neq \emptyset$ and a point $y \in V \cap X_n$ having an element $B \in T(y, X_n) \subset C(X_n)$ with $H(A, B) < \varepsilon$ for all $n > N$. Thus $A \in \text{LiC}(X_n)$ and hence $C(X_0) \subset \text{LiC}(X_n)$. Therefore we have $C(X_0) = \text{Ltc}(X_n)$.

**Theorem 2.10.** The sequence $\{X_n\}_{n=1}^{\infty}$ of subcontinua of a metric continuum $X$ converges to $X_0$ w-regularly if and only if the sequence $\{C(X_n)\}_{n=1}^{\infty}$ converges to $C(X_0)$.

**Proof.** Suppose $X_n \to X_0$ is w-regular. It suffices to show that $C(X_0) \subset \text{LiC}(x_n)$. Let $A \in C(X_0)$, and $\varepsilon > 0$ be given. Let $a \in A$. The w-regular convergence implies that there is $\delta > 0$ and $N$ such that the $\delta$-neighborhood $V$ of $a$ intersects $X_n$ for all $n > N$ and there is a point $a_n \in X_n$ having an element $A_n \in T(a_n, X_n)$ such that $H(A, A_n) < \varepsilon$ for each $n \geq N$. Thus $A \in \text{LiC}(x_n)$. Hence we have $\text{Ltc}(X_n) = C(X_0)$.

Now suppose $\text{Ltc}(X_n) = C(X_0)$. Let $\varepsilon > 0$ and $A \in T(a, X_0)$. Let $\{A_n\}_{n=1}^{\infty}$, $A_n \in C(X_n)$, be a sequence which converges to $A$. Let $N$ be an integer such that $H(A, A_n) < \varepsilon$ for all $n \geq N$. Let $\delta = \varepsilon$, and let $V$ be the $\delta$-neighborhood of $a$. Then $V \cap A_n \neq \emptyset$ for all $n \geq N$. So we pick a point $a_n \in A_n$ for each $n \geq N$. Then these satisfy w-regular convergence condition.

3. $R^i$-continua in $C(X)$

In [1], it was proven that if a metric space $X$ contains a proper subset $A$ which is homotopically fixed, then $X$ is not contractible. Subsequently Czuba [4] proved that any $R^i$-continua of a dendroid is homotopically fixed. But it can be verified that it holds for all metric continua. In [2], there were some attempts to generalize $R^i$-continua of dendroids for metric continua $X$ [2, Proposition 1] and extending them to hyperspaces $C(X)$ [2, Theorem 6, Corollary 7, and Corollary 17]. (The statement of Corollary 17 remains true by [8]).

In this section, we will remedy the attempts for a subclass of metric continua.

The following definition was originally given for the class of dendroid.
**Definition 3.1** [3]. Let $X$ be a metric continuum. A nonempty proper subcontinuum $K$ of $X$ is called

1. an $R^1$-continuum if there exists an open set $U$ such that $K \subset U$ and two sequences $\{C_n^1\}_{n=1}^{\infty}$ of components of $U$ such that $K = LsC_n^1 \cap LsC_n^2$;
2. an $R^2$-continuum if there exist an open set $U$ containing $K$ and two sequences $\{C_n^1\}_{n=1}^{\infty}$, $\{C_n^2\}_{n=1}^{\infty}$ of components of $U$ such that $K = LtC_n^1 \cap LtC_n^2$;
3. an $R^3$-continuum if there exists an open set $U$ containing $K$ and a sequence $\{C_n\}_{n=1}^{\infty}$ of components of $U$ such that $K = LiC_n$.

In sequel, we denote $R^1$-continuum, $R^2$-continuum, and $R^3$-continuum by $LsC_n^1 \cap LsC_n^2 \subset U$, $LtC_n^1 \cap LtC_n^2 \subset U$, and $LiC_n \subset U$, respectively, as the open set $U$ and the components are given in the definition.

Now Czuba’s Proposition 5 and a part of Corollary 11 in [3] can be stated for metric continua but his Proposition 10 in [3] can not be generalized for metric continua (see Example C below).

**Proposition 3.2** [3].

(a) Each $R^2$-continuum of a metric continuum $X$ is both $R^1$ and $R^3$-continuum.

(b) If $R^1$-continuum of a metric continuum $X$ is a single point, then it is both $R^2$ and $R^3$-continuum.

**Proof.**

(a) In fact, if $K = LtC_n^1 \subset U$, then $LsC_n^i = LtC_n^i$ for each $i = 1, 2$, so that $K$ is an $R^1$-continuum. Now define a new sequence $\{D_n\}_{n=1}^{\infty}$ by letting $D_{2n} = C_n^1$ and $D_{2n+1} = C_n^2$. Then it is easy to check that $K = LiD_n \subset U$.

(b) Suppose $K = \{x\} = LsC_n^1 \cap LsC_n^2 \subset U$. For each $i = 1, 2$, choose a convergent subsequence $\{C_{n_k}^i\}_{k=1}^{\infty}$ of $\{C_n^i\}_{n=1}^{\infty}$ whose limit contains $x$. Then $LtC_{n_k}^i \subset LsC_n^i$ for each $i = 1, 2$, implies that $LtC_{n_k}^1 \cap LtC_{n_k}^2$. The proof that $K = \{x\}$ is an $R^3$-continuum is the same as in (a).
W-regular convergence of $R^i$-continua

**Theorem 3.3.** Let $K = LtC^1_n \cap LtC^2_n \subset U$ be an $R^2$-continuum of a metric continuum $X$ such that the convergence of each sequences $\{C^i_n\}_{n=1}^{\infty}$, $i = 1, 2$, is $w$-regular. Then $C(K)$ is a $R^2$-continuum in $C(X)$.

**Proof.** Since $C(U)$ is open in $C(X)$ and each $C(C^i_n)$ is a component of $C(U)$, we let $\mathcal{K} = LtC(C^1_n) \cap LtC(C^2_n)$. Let $K_i = LtC^i_n$ for each $i = 1, 2$. Then $K* \subset \mathcal{K}$ so that $\mathcal{K} \neq \emptyset$.

Since the convergence is $w$-regular, we have $C(K_i) = LtC(C^i_n)$ for each $i$ by Theorem 2.9. Let $A \in \mathcal{K}$. Then $A \subset C(K_1) \cap C(K_2)$ so that $A \in C(K)$. On the other hand, suppose $A \in C(K)$. Then $A \subset K_1 \cap K_2$ so that $A \in C(K_1) \cap C(K_2)$. This shows that $C(K) = \mathcal{K}$. Since $C(K)$ is connected, it is an $R^2$-continuum.

**Theorem 3.4.** $K = LsC^1_n \cap LsC^2_n \subset U$ be an $R^1$-continuum of a metric continuum $X$ having the property that each converging subsequence of $\{C^i_n\}_{n=1}^{\infty}$, $i = 1, 2$, converges $w$-regularly. Then $\mathcal{K} = LsC(C^1_n) \cap LsC(C^2_n)$ is an $R^1$-continuum of $C(X)$.

**Proof.** Since the continuum $K*$ is contained in $\mathcal{K}$, $\mathcal{K}$ is nonempty and compact. We show that $\mathcal{K}$ is connected. Let $A \in \mathcal{K}$. Then, for each $i = 1, 2$, there is a sequence $\{A^i_{n_k}\}_{k=1}^{\infty}$, $A^i_{n_k} \in C(C^i_{n_k})$, such that $A^i_{n_k} \to A$. Then $A \subset LiC^i_{n_k}$ for each $i = 1, 2$. Let $D^i_j \supseteq \{D^i_{n_k}\}_{j=1}^{\infty}$ be convergent subsequence of $\{C^i_{n_k}\}_{k=1}^{\infty}$ for each $i = 1, 2$. Then $A \subset LiD^i_j = LsD^i_j$ for each $i = 1, 2$. Since the convergence is $w$-regular, $C(A) \subset LtC(D^1_j) \cap LtC(D^2_j) \subset LsC(C^1_n) \cap LsC(C^2_n)$. Thus the connected set $K* \cup C(A)$ is contained in $\mathcal{K}$. Therefore, $\mathcal{K}$ is connected and hence is an $R^1$-continuum of $C(X)$.

**Remark.** In Theorem 3.4, we can not say that $\mathcal{K} = C(K)$. In fact, most likely $K \notin \mathcal{K}$ (see [3, Example 4] or Example C below).

**Theorem 3.5.** Let $K = LiC_n \subset U$ be an $R^3$-continuum of a metric continuum $X$ with the property that each converging subsequence of $\{C_n\}_{n=1}^{\infty}$ converges $w$-regularly. Then $\mathcal{K} = LiC(C_n)$ is an $R^3$-continuum of $C(X)$.

**Proof.** Let $\mathcal{K} = LiC(C_n)$. Clearly $K* = \{x \in X\} \subset \mathcal{K}$. Let $A \in \mathcal{K}$ and let $\{A_n\}_{n=1}^{\infty}$, $A_n \in C(C_n)$, be a sequence which converges to $A$. Then clearly $A \in C(K)$. We show first that $C(A) \subset LsC(C_{n_k})$
for each subsequence \( \{C(C_{n_k})\}_{k=1}^{\infty} \) of \( \{C(C_n)\}_{n=1}^{\infty} \). So let \( \{C_{n_k}\}_{k=1}^{\infty} \) be any subsequence of \( \{C_n\}_{n=1}^{\infty} \). Since \( K \subset LsC_{n_k} \), we have \( A \subset LsC_{n_k} \). Let \( \{D_j\}_{j=1}^{\infty} \) be an convergent subsequence of \( \{C_{n_k}\}_{k=1}^{\infty} \). Then \( A \subset LtD_j \). Since the convergence is \( w \)-regular by our assumption, we have \( A \subset C(LtD_j) = LtC(D_j) \). Now \( C(A) \subset C(LtD_j) \) and \( LtC(D_j) \subset LsC(C_{n_k}) \). Hence \( C(A) \subset LsC(C_{n_k}) \) for every subsequence \( \{C_{n_k}\}_{k=1}^{\infty} \). Therefore, \( C(A) \subset K \) by [5].

Since \( K^* \cup C(A) \) is connected and contained in \( K \) for each \( A \in K \), \( K \) is connected. This proves that \( K \) is an \( R^3 \)-continuum.

**EXAMPLE C (W.J. Charatonik).** We give an example of an \( R^1 \)-continuum of a metric continuum \( X \) which contains neither \( R^2 \)-continuum nor \( R^3 \)-continuum. This is a modified version of the example recently given by W.J. Charatonik to one of the authors.

We construct the example in \( E^3 \). If \( p,q \in E^3 \), the straight line segment between \( p \) and \( q \) is denoted by \( pq \). Let \( a = (2,0,0) \), \( q_0 = (1,1,0) \), \( r_0 = (-1,1,0) \), \( s_0 = (-1,-1,0) \) and \( t_0 = (1,-1,0) \). For each positive integer \( n \), let \( p_n^+ = (\frac{n+2}{n+1}, \frac{1}{n+1}, 0) \), \( p_n^- = (\frac{n+2}{n+1}, \frac{-1}{n+1}, 0) \), \( q_n^+ = (\frac{n+2}{n+1}, \frac{n+2}{n+1}, 0) \), \( q_n^- = (\frac{n}{n+1}, \frac{n+2}{n+1}, 0) \), \( r_n^+ = (\frac{-n(n+2)}{n+1}, \frac{n+2}{n+1}, 0) \), \( r_n^- = (\frac{-n(n+2)}{n+1}, \frac{n}{n+1}, 0) \), \( s_n^+ = (\frac{-n(n+2)}{n+1}, \frac{-n(n+2)}{n+1}, 0) \), \( s_n^- = (\frac{-n(n+2)}{n+1}, \frac{-n}{n+1}, 0) \), \( t_n^+ = (\frac{n+2}{n+1}, \frac{-n(n+2)}{n+1}, 0) \), \( t_n^- = (\frac{n+2}{n+1}, \frac{-n}{n+1}, 0) \).

Let \( S_0 = q_0r_0 \cup s_0t_0 \cup t_0q_0 \), and \( S_1 = S_0 \cup r_0s_0 \). For each positive integer \( n \), considering an ordering in \( F_n = \{a, p_n^+, q_n^+, r_n^+, t_n^+, s_n^+, s_n^-, t_n^-, p_n^-, a\} \). Let \( D_n \) be the union of all line segments between two consecutive elements as listed in \( F_n \). Let \( H_0 = \cup_{n=1}^{\infty} D_n \), and let \( X_0 = S_0 \cup H_0 \). Let \( f : E^3 \rightarrow E^3 \) be the rotation about the line \( x = y = 0 \) by the angle \( \frac{\pi}{2} \). Let \( X_i = f^i(X_0) \), \( (f^0 = \text{identity}, f^2 = \text{the composition of} f \text{ and } f, \text{ etc.)}, \) for \( i = 0,1,2,3 \). Let raise \( X_i \) straight upward to the \( z = i \) plane for \( i = 1,2,3 \), so that the resulting continua \( \hat{X_i}, i = 1,2,3 \), together with \( X_0 = \hat{X}_0 \) are pairwise disjoint.

We now wish to identify only those segments of \( f^i(S_0), i = 1,2,3 \), in the \( z = i \) plane to the those of \( S_1 \) so that \( f^i(H_0) \) are pairwise disjoint. The identifying relation \( \sim \) is defined as follows; let \( x \in r_0q_0 \), \( y \in f(q_0)f(t_0) \), and \( z \in f^2(t_0)f^2(s_0) \). Then the three points \( x, y, \) and \( z \) are identified as one point if and only if their first and second coordinates are the same. For each of other three triplets of segments,
W-regular convergence of $R^i$-continua

$\{q_0t_0, f(t_0)f(s_0), f^2(r_0)f^3(q_0)\}, \{s_0t_0, f^2(q_0)f^i(r_0), f^3(t_0)f^3(q_0)\}$, and $\{f(q_0)f(r_0), f^2(t_0)f^2(q_0), f^3(s_0)f^3(t_0)\}$, we identify in the same way. Now let $X = (S_0 \cup X_0 \cup \bigcup_{i=1}^3 \tilde{X}_i) / \sim$ be the quotient space. Then the only $R^i$-continuum in $X$ is $S_1$ (which is homeomorphic to the unit circle). Furthermore, $S_1$ is an $R^1$-continuum of $X$ which contains neither $R^2$-continuum nor $R^3$-continuum.

**Example D.** Let $X$ be the space in Example C and let $Y$ be the space in the Example 4 in [3]. Let $Z = (X \cup Y)/\{a, x\}$ be the quotient space, where $a$ is the point in $X$ and $x$ is a point in $Y$ at which $Y$ is locally connected. Then $Z$ contains the circle $S_1$ so that $Z$ is not a dendroid. Also $Z$ contains $S_1$ as an $R^1$-continuum, the segment $ee'$ in $Y$ as an $R^2$-continuum, and the singleton subset $\{q\}$ of $Y$ as an $R^3$-continuum.

**References**


**Department of Mathematics, Wayne State University, Detroit, MI 48202, U.S.A.**

**Department of Mathematics, Chonbuk National University, Chonju, Chonbuk 560-756, Korea**

**Department of Mathematics, Chunju University, Chonju, Chonbuk 560-759, Korea**