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THE BERGMAN KERNEL FUNCTION AND THE DENSITY THEOREMS IN THE PLANE

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1. Introduction

The Bergman kernel is closely connected to mapping problems in complex analysis. For example, the Riemann mapping function is written down in terms of the Bergman kernel. Hence, information about the Bergman kernel gives information about mappings. In this note, we prove the following theorem.

THEOREM 2. *If Ω is a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary and $w_0 \in b\Omega$, then for any smoothly bounded simply-connected domain $\Omega_0 \subset \Omega$ with $w_0 \notin b\Omega_0$, span$\{\partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$ is dense in $A^\infty(\Omega_0)$.*

It is an improvement in the plane of a theorem by Klingenberg [7]. As a corollary to this density theorem, we get that for any $w_0, z_0 \in b\Omega$ with $w_0 \neq z_0$, the associated Bergman kernel $K(z_0, w)$ vanishes to finite order as $w \to w_0$. The question of non-vanishing to infinite order in w of $K(z_0, w)$ arises in connection with the problem of boundary behavior of biholomorphic mappings and we conjecture that this corollary is true for any smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ of finite type (see [2]). Theorem 4 is another density theorem obtained by using Theorem 2. We get the similar results for the Szegő kernel in section 4.

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2. Notation and Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary and let $b\Omega$ denote the boundary of Ω.

Let s be a non-negative integer.

$W^s(\Omega)$ is the usual Sobolev space of the complex valued functions on Ω with the inner product given by

$$\langle u, v\rangle_{s,\Omega} = \sum_{|\alpha|\le s} \int_\Omega D^\alpha u\, \overline{D^\alpha v}\, dV.$$

The norm $\|\cdot\|_{s,\Omega}$ of $W^s(\Omega)$ induced by $\langle , \rangle_{s,\Omega}$ is $\|u\|_{s,\Omega} = \langle u, u\rangle_{s,\Omega}^{1/2}$ for $u \in W^s(\Omega)$.

$W_0^s(\Omega)$ is the closure in $W^s(\Omega)$ of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω.

$W^{-s}(\Omega)$ is the dual space of $W_0^s(\Omega)$ and is identified with the space of distributions λ such that

$$\|\lambda\|_{-s,\Omega} = \sup\,\{|\lambda(\varphi)| : \varphi \in C_0^\infty(\Omega), \|\varphi\|_{s,\Omega} = 1\}$$

is finite.

The space $C^\infty(\overline{\Omega})$ of complex valued functions which are smooth up to the boundary of Ω is the intersection of the spaces $W^s(\Omega)$ by Sobolev's lemma. The intersection $W_0^\infty(\Omega)$ of the spaces $W_0^s(\Omega)$ is the space of smooth functions vanishing to infinite order at the boundary. $C^\infty(\overline{\Omega})$ and its closed subspace $W_0^\infty(\Omega)$ are Fréchet spaces with projective limit topologies. On the other hand, $W^{-\infty}(\Omega)$ is the union of the spaces $W^{-s}(\Omega)$ with inductive limit topology.

$A(\Omega)$ is the set of holomorphic functions in Ω. Denote by $A^\infty(\Omega)$ the set $A(\Omega) \cap C^\infty(\overline{\Omega})$, and by $A^{-\infty}(\Omega)$ the holomorphic subspace of $W^{-\infty}(\Omega)$.

Let $K(z, w)$ be the Bergman kernel function associated to Ω. It is holomorphic in z and anti-holomorphic in w. It is conjugate-symmetric, i.e. $K(z, w) = \overline{K(w, z)}$. Let $K_1(z, w)$ and $K_2(z, w)$ denote the Bergman kernels associated to Ω_1 and Ω_2, respectively. The Bergman kernels transform according to the formula

$$K_1(z, w) = f'(z)K_2(f(z), f(w))\overline{f'(w)} \qquad z, w \in \Omega_1$$

when $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between bounded domains. See [8] for the elementary properties of the Bergman kernel. Note that $K(z,w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in b\Omega\})$ (see [3; p.100]).

Here we use the following notations:

$$\partial_z^i K(z_0, w) = \frac{\partial^i K(z,w)}{\partial z^i}|_{z=z_0}, \qquad \partial_{\bar{w}}^k K(z, w_0) = \frac{\partial^k K(z,w)}{\partial \bar{w}^k}|_{w=w_0}.$$

3. Main Results

Klingenberg [7] proved that if $\Omega \subset \mathbb{C}^n$ is a strictly pseudoconvex domain with real analytic boundary and $w_0 \in b\Omega$, then for any $z_0 \in b\Omega - \{w_0\}$, $\text{span}\{\partial_{\bar{w}}^\alpha K(\cdot, w_0) : |\alpha| \geq 0\}$ is dense in $A(\Omega_0) \cap W^s(\Omega_0)$ for all $s \geq 0$ and some domain $\Omega_0 \subset \Omega$ with $z_0 \in b\Omega_0$ and $w_0 \notin b\Omega_0$. To get an improvement of it in the plane, we need the following lemma.

LEMMA 1. *Let Ω be a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary. For any smoothly bounded simply-connected domain $\Omega_0 \subset \Omega$, $A^\infty(\Omega)_{|\Omega_0}$ is dense in $A^\infty(\Omega_0)$.*

Proof. Let Ω_0 be given and ρ_0 be the defining function for Ω_0, i.e., $\Omega_0 = \{z : \rho_0(z) < 0\}$. For any $f \in A^\infty(\Omega_0)$, there exists an open neighborhood U of $\overline{\Omega}_0$ so that f is the restriction to $\overline{\Omega}_0$ of a C^∞ function F on U. Write $\Omega_\delta = \{z \in U : \rho_0(z) < \delta\}$. Let $s \geq 0$ and $\varepsilon > 0$ be given. There exists a $\delta_0 = \delta_0(s, \varepsilon)$ so that $\|\partial_{\bar{z}} F\|_{s,\Omega_\delta} < \varepsilon$ whenever $\delta < \delta_0$. Let G_δ denote the Green's operator associated to Ω_δ. Then $\Lambda_\delta = 4\partial_z G_\delta$ satisfies that $\partial_{\bar{z}}(\Lambda_\delta \partial_{\bar{z}} F) = \partial_{\bar{z}} F$ on Ω_δ and

$$\|\Lambda_\delta \partial_{\bar{z}} F\|_{s,\Omega_\delta} \leq C \|\partial_{\bar{z}} F\|_{s-1,\Omega_\delta}$$

with C independent of δ for $\delta < \delta_0$ by standard elliptic theory (see [5; p.177]). Therefore, $F - \Lambda_\delta \partial_{\bar{z}} F \in A^\infty(\Omega_\delta)$ and

$$\begin{aligned} \|f - (F - \Lambda_\delta \partial_{\bar{z}} F)\|_{s,\Omega_0} &= \|\Lambda_\delta \partial_{\bar{z}} F\|_{s,\Omega_0} \\ &\leq C \|\partial_{\bar{z}} F\|_{s-1,\Omega_\delta}. \end{aligned}$$

Take $\delta < \delta_0$ small enough that $C\|\partial_{\bar z}F\|_{s-1,\Omega_\delta} < \varepsilon$. Set $f_\delta = F - \Lambda_\delta \partial_{\bar z} F$.

By Runge's approximation theorem, there exists a polynomial g with

$$\sup_{\overline{\Omega}_{\delta/2}} |g - f_\delta| < \frac{\varepsilon}{\left(\sum_{0\le k\le s}(\frac{k!}{d^k})^2 \mathrm{Vol}(\Omega_0)\right)^{\frac{1}{2}}}$$

where $d = \mathrm{dist}(\Omega_0, \Omega^c_{\delta/2})$. Let $h = g - f_\delta \in A^\infty(\Omega_\delta)$. By Cauchy's integral formula, for $z \in \Omega_0$

$$\partial_z^k h(z) = \frac{k!}{2\pi i}\int_{bB(z;d)} \frac{h(w)}{(w-z)^{k+1}}\, dw$$

where $bB(z;d) = \{w : |z-w| = d\}$. It follows that $|\partial_z^k h(z)| \le \frac{k!}{d^k} M$ where $M = \sup_{bB(z;d)} |h| \le \sup_{\overline{\Omega}_{\delta/2}} |g - f_\delta|$. Therefore,

$$\|h\|^2_{s,\Omega_0} = \sum_{0\le k\le s} \int_{\Omega_0} |\partial_z^k h(z)|^2 \le \sum_{0\le k\le s} \left(\frac{k!}{d^k}\right)^2 M^2\, \mathrm{Vol}(\Omega_0) < \varepsilon^2.$$

Hence $\|f - g\|_{s,\Omega_0} \le \|f - f_\delta\|_{s,\Omega_0} + \|f_\delta - g\|_{s,\Omega_0} < 2\varepsilon$. □

We are ready to have the following density theorem by using Lemma 1.

THEOREM 2. *Let Ω be a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary and let $w_0 \in b\Omega$. For any smoothly bounded simply-connected domain $\Omega_0 \subset \Omega$ with $w_0 \notin b\Omega_0$,*

$$span\{\partial_{\bar w}^k K(\cdot, w_0) : k \ge 0\}$$

is dense in $A^\infty(\Omega_0)$.

Proof. Assume that the conclusion is not true for a smoothly bounded simply-connected domain $\Omega_0 \subset \Omega$ with $w_0 \notin b\Omega_0$. Then there exists a non-zero $g \in A^{-\infty}(\Omega_0)$ such that $\langle \partial_{\bar w}^k K(\cdot, w_0), g\rangle_{0,\Omega_0} = 0$ for each $k \ge 0$ since the dual space of $A^\infty(\Omega_0)$ is equal to $A^{-\infty}(\Omega_0)$ (see [1] [3; p.122]). Now, let $h(w) = \langle g, K(\cdot, w)\rangle_{0,\Omega_0}$ for $w \in \Omega$. Then $\partial_w^k h(w_0) = 0$ for each $k \ge 0$.

Let $f : \Omega \to \tilde{\Omega}$ be a biholomorphic mapping from Ω to $\tilde{\Omega}$ where $\tilde{\Omega}$ is a bounded domain with real analytic boundary having same connectivity as Ω (see [9; p.341]). Note that f extends smoothly up to $b\Omega$ and $(f^{-1})' \cdot (h \circ f^{-1})$ vanishes to infinite order at $f(w_0)$. Let $\tilde{K}(\cdot,\cdot)$ be the Bergman kernel function associated to $\tilde{\Omega}$. Observe that $\tilde{K}(z,\zeta) = -\frac{2}{\pi}\partial_z \partial_{\bar{\zeta}} G(z,\zeta)$ and so $\tilde{K}(z,\zeta)$ extends anti-holomorphically past $b\tilde{\Omega}$ since Green's function $G(z,\zeta)$ for $\tilde{\Omega}$ extends harmonically in ζ to a neighborhood of $f(w_0)$ for $z \in \tilde{\Omega}$.

By the transformation formula for the Bergman kernels,

$$\begin{aligned}(f^{-1})'(\zeta) \cdot h(f^{-1}(\zeta)) &= \langle g, K(\cdot, f^{-1}(\zeta))\overline{(f^{-1})'(\zeta)}\rangle_{0,\Omega_0} \\ &= \langle g, f'(\cdot)\tilde{K}(f(\cdot),\zeta)\rangle_{0,\Omega_0}.\end{aligned}$$

Hence $(f^{-1})' \cdot (h \circ f^{-1})$ extends holomorphically past $b\tilde{\Omega}$ near $f(w_0)$. It implies that $(f^{-1})' \cdot (h \circ f^{-1}) \equiv 0$ on $\tilde{\Omega}$ and so $h \equiv 0$ on Ω. Hence g is orthogonal to $\text{span}\{K(\cdot,w) : w \in \Omega\}_{|\Omega_0}$. Since the Bergman projection maps $C^\infty(\overline{\Omega})$ into itself (see [3; p.60]), $\text{span}\{K(\cdot,w) : w \in \Omega\}$ is dense in $A^\infty(\Omega)$ by [4]. Note that $A^\infty(\Omega)_{|\Omega_0}$ is dense in $A^\infty(\Omega_0)$ by Lemma 1. So $\text{span}\{K(\cdot,w) : w \in \Omega\}_{|\Omega_0}$is dense in $A^\infty(\Omega_0)$. Therefore g is orthogonal to $A^\infty(\Omega_0)$ and $g \equiv 0$. It is a contradiction. □

REMARK. Let Ω be the unit disc in $\mathbb{C}$. It is worth evaluating the derivatives of the Bergman kernel function explicitly. Indeed,

$$K(z,w) = \frac{1}{\pi(1 - z\bar{w})^2} \qquad \text{on } \Omega.$$

For fixed $w_0 \in b\Omega$,

$$\text{span}\{\partial^i_{\bar{w}} K(z,w_0) : i \geq 0\} = \text{span}\{\frac{1}{(z - w_0)^i} : i \geq 2\}.$$

It is easy to see by direct calculation that for any smoothly bounded simply-connected domain $\Omega_0 \subset \Omega$ with $w_0 \notin b\Omega$, $\text{span}\{\frac{1}{(\cdot - w_0)^i} : i \geq 2\}$ is dense in $A^\infty(\Omega_0)$.

The above theorem implies

COROLLARY 3. *Let* $w_0 \in b\Omega$. *For any* $z_0 \in b\Omega - \{w_0\}$, $K(z_0, w)$ *vanishes to finite order as* $w \to w_0$.

Proof. Let $z_0 \in b\Omega - \{w_0\}$ and $\Omega_0 \subset \Omega$ be any smoothly bounded simply-connected domain with $z_0 \in b\Omega_0$ and $w_0 \notin b\Omega_0$. By Theorem 2, $\text{span}\{\partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$ is dense in $A^\infty(\Omega_0)$. Since $1 \in A^\infty(\Omega_0)$, for $\varepsilon > 0$, there exist a non-negative integer $m = m(\varepsilon)$ and the constant coefficients $c_0, \ldots, c_m$ such that

$$\|1 - \sum_{k=0}^{m} c_k \, \partial_{\bar{w}}^k K(\cdot, w_0)\|_{1,\Omega_0} < \varepsilon.$$

By Sobolev's lemma

$$\begin{aligned} &\sup_{z \in \overline{\Omega}_0} |1 - \sum_{k=0}^{m} c_k \, \partial_{\bar{w}}^k K(z, w_0)| \\ &\leq C \, \|1 - \sum_{k=0}^{m} c_k \, \partial_{\bar{w}}^k K(\cdot, w_0)\|_{1,\Omega_0} \end{aligned}$$

for a constant C. Take ε small enough to satisfy $C\varepsilon < 1$. Hence, for $z_0 \in b\Omega - \{w_0\}$, there exists a k_0 with $0 \leq k_0 \leq m$ such that $\partial_{\bar{w}}^{k_0} K(z_0, w_0) \neq 0$. □

We get another density theorem by using Theorem 2.

THEOREM 4. *Let* Ω *be a bounded domain in* $\mathbb{C}$ *with* C^∞ *smooth boundary and let* $w_0 \in b\Omega$. *Suppose that* Ω_0 *is any smoothly bounded simply-connected domain in* Ω *with* $w_0 \notin b\Omega_0$. *Then, for each* $i \geq 0$,

$$\text{span}\{\partial_z^i \partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$$

is dense in $A^\infty(\Omega_0)$.

Proof. For given Ω_0, $\text{span}\{\partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$ is dense in $A^\infty(\Omega_0)$ by Theorem 2. Let $i \geq 0$ be given and $f \in A^\infty(\Omega_0)$. For each $s \geq 0$ and $\varepsilon > 0$, there exist a non-negative integer $m = m(s+i, \varepsilon)$ and the constant coefficients $c_0, \ldots, c_m$ such that

$$\|\sum_{k=0}^{m} c_k \partial_{\bar{w}}^k K(\cdot, w_0) \; - \; f\|_{s+i,\Omega_0} < \varepsilon.$$

Hence

$$\|\sum_{k=0}^{m} c_k \partial_z^i \partial_{\bar{w}}^k K(\cdot, w_0) - \partial_z^i f\|_{s,\Omega_0} < \varepsilon$$

and so $\text{span}\{\partial_z^i \partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$ is dense in $\partial_z^i A^\infty(\Omega_0)$ where $\partial_z^i A^\infty(\Omega_0) = \{\partial_z^i f : f \in A^\infty(\Omega_0)\}$. For each $g \in A^\infty(\Omega_0)$, fix $a \in \Omega_0$ and take $f_1(z) = \int_{\gamma_a^z} g(z)\, dz$ where $\gamma_a^z \subset \Omega_0$ is a polygonal path from a to z. Then $f_1 \in A^\infty(\Omega_0)$ and $\partial_z f_1(z) = g(z)$. So $\partial_z A^\infty(\Omega_0) = A^\infty(\Omega_0)$. Repeat it to get $\partial_z^i A^\infty(\Omega_0) = A^\infty(\Omega_0)$. Thus,

$$\text{span}\{\partial_z^i \partial_{\bar{w}}^k K(\cdot, w_0) : k \geq 0\}$$

is dense in $A^\infty(\Omega_0)$. □

Theorem 4 implies that for any $w_0, z_0 \in b\Omega$ with $w_0 \neq z_0$ and any $i \geq 0$, $\partial_z^i K(z_0, w)$ vanishes to finite order as $w \to w_0$.

4. The Szegő Kernel

Let Ω be a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary $b\Omega$.

Let $C^\infty(b\Omega)$ denote the space of functions defined on $b\Omega$ which are C^∞ smooth. For u and v in $C^\infty(b\Omega)$, the L^2 inner product on $b\Omega$ of u and v is defined by $\langle u, v\rangle_{b\Omega} = \int_{b\Omega} u\bar{v}\, ds$. $L^2(b\Omega)$ is the Hilbert space completion of $C^\infty(b\Omega)$ with respect to this inner product. $H^2(b\Omega)$ is the closed subspace of $L^2(b\Omega)$ of boundary values of holomorphic functions on Ω.

Let $S(z, w)$ denote the Szegő kernel associated to Ω. It is holomorphic in z and anti-holomorphic in w. It is conjugate-symmetric. Note that $S(z, w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in b\Omega\})$ and the Szegő kernel associated to a domain with real analytic boundary extends holomorphically past the boundary. Also, $\text{span}\{S(\cdot, w) : w \in \Omega\}$ is dense in $A^\infty(\Omega)$. See [3] for the properties of the Szegő kernel.

Let $S_1(z, w)$ and $S_2(z, w)$ denote the Szegő kernels associated to Ω_1 and Ω_2, respectively. Observe that the Szegő kernels transform according to the formula

$$S_1(z, w) = \sqrt{f'(z)} S_2(f(z), f(w)) \overline{\sqrt{f'(w)}} \qquad z, w \in \Omega_1$$

when $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between two smoothly bounded n-connected domains in $\mathbb{C}$ (see [3; p.44], [6]).

Indeed, it is easy to see that f' has a single-valued square root on Ω_1. Let $\sqrt{f'}$ denote one of the square root of f'. The arc-length magnification from $b\Omega_1$ to $b\Omega_2$ is $|f'(z)|$. If $\{\psi_j\}$ is a complete orthonormal system on $H^2(b\Omega_2)$, then so is $\{\sqrt{f'(z)}\psi_j(f(z))\}$ on $H^2(b\Omega_1)$. Hence

$$\begin{aligned} S_1(z,w) &= \sum_{j=1}^{\infty} \sqrt{f'(z)}\psi_j(f(z))\overline{\sqrt{f'(w)}}\,\overline{\psi_j(f(w))} \\ &= \sqrt{f'(z)} \sum_{j=1}^{\infty} \psi_j(f(z))\overline{\psi_j(f(w))}\overline{\sqrt{f'(w)}} \\ &= \sqrt{f'(z)} S_2(f(z), f(w))\overline{\sqrt{f'(w)}} \qquad z \in \Omega_1,\ w \in b\Omega_1. \end{aligned}$$

With the identification of functions in $H^2(b\Omega_1)$ with their unique holomorphic extensions to Ω_1, it follows that

$$S_1(z,w) = \sqrt{f'(z)} S_2(f(z), f(w))\overline{\sqrt{f'(w)}}$$

for z, $w \in \Omega_1$.

Now we get the conclusion similar to Theorem 2 for the Szegő kernel by modifying the proof of Theorem 2. Finally we have the following result similar to Theorem 4.

PROPOSITION 5. *Let Ω be a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary and let $w_0 \in b\Omega$. Suppose Ω_0 is any smoothly bounded simply-connected domain in Ω with $w_0 \notin b\Omega_0$. Then, for each $i \geq 0$,*

$$span\{\partial_z^i \partial_{\bar{w}}^k S(\cdot, w_0) : k \geq 0\}$$

is dense in $A^\infty(\Omega_0)$.

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