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SOME PROPERTIES OF CERTAIN NONHYPONORMAL OPERATORS

SANG HUN LEE AND CHEON SEOUNG RYOO

1. Introduction

Let $\mathcal{H}$ be an arbitrary complex Hilbert space and let T be a bounded linear operators on $\mathcal{H}$. The $*$-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $L(\mathcal{H})$ and $K(\mathcal{H})$ is the ideal of all compact operators on $\mathcal{H}$. The spectrum, the point spectrum, the approximate point spectrum and continuous spectrum of an operator T are denoted by $\sigma(T), \sigma_p(T), \sigma_{ap}(T), \sigma_c(T)$, respectively. The quotient algebra $L(\mathcal{H})/K(\mathcal{H})$ is called the Calkin algebra. Let $\pi : L(\mathcal{H}) \to L(\mathcal{H})/K(\mathcal{H})$ be the natural mapping. Then an operator T is said to be Fredholm if $\pi(T)$ is invertible. The essential spectrum of T is the set of all complex number λ such that $\lambda I - T$ is not Fredholm operator and is denoted by $\sigma_e(T) = \sigma(\pi(T))$. Let $\mathcal{A}$ be a commutative Banach algebra with a unit element and let Φ be the maximal ideal space of $\mathcal{A}$. For each a in $\mathcal{A}$, define $\hat{a} : \Phi \to \mathbf{C}$ by $\hat{a}(h) = h(a)$ for each $h \in \Phi$. Then the functional $\hat{a}$ is called the Gel'fand transform of a and it is continuous. This defines a mapping $\rho : \mathcal{A} \to C(\Phi)$ by $\rho(a) = \hat{a}$, where $C(\Phi)$ is the set of all continuous complex-valued functions on Φ.

This mapping is a homomorphism and is called the Gel'fand transformation of $\mathcal{A}$. If a is in $\mathcal{A}$, then $\sigma(a) = \{h(a) : h \in \Phi\} = \hat{a}(\Phi)$. Hence, $\|\hat{a}\|_\infty = \sup\{|\hat{a}(h)| : h \in \Phi\}$: the spectral radius of a. Thus $\|\hat{a}\|_\infty < \|a\|$.

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DEFINITION 1.1. [8]. T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector x in $\mathcal{H}$. This class of operators was introduced and studied by Patel [12] under the title operators of class (M). Clearly, the class of $*$-paranormal operators includes the class of hyponormal operators and also this class is closed in the norm topology of operators. Furthermore this class is independent of the class of paranormal operators.

Also an operator T is $*$-paranormal if and only if $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$ for each $\lambda > 0$.

In this paper we will investigate some properties of $*$-paranormal operators and study the images of $*$-representation of $L(\mathcal{H})$ on $\mathcal{K}$.

2. Some results

In this section, we will study some property which are applied for the hyponormal operators.

PROPOSITION 2.1. *Let T be a $*$-paranormal operator.*

(1) *If $\lambda \in \sigma_p(T)$, then $\bar{\lambda} \in \sigma_p(T^*)$.*
(2) *If $Tx = \lambda x$, and $Ty = \mu y$ and $\lambda \neq \mu$, then $< x, y >= 0$.*
(3) *If M and $M^\perp \in LatT$, then $T|M$ and $T|M^\perp$ are $*$-paranormal.*
(4) *If $\lambda \in \sigma_{ap}(T)$, then $\bar{\lambda} \in \sigma_{ap}(T^*)$.*

Proof. (1) If $\lambda = 0$, the result is obvious, we may assume that $\lambda \neq 0$, since

$$\|T^*x\|^2 \leq \|T^2x\| = |\lambda|^2\|x\| = |\lambda|^2.$$

Thus $\|T^*x\|^2 \leq |\lambda|^2$ and

$$\begin{aligned}\|T^*x - \bar{\lambda}x\|^2 &=< T^*x - \bar{\lambda}x, T^*x - \bar{\lambda}x > \\ &=< T^*x, T^*x > -\bar{\lambda} < x, T^*x > -\lambda < T^*x, x > +|\lambda|^2 \\ &= \|T^*x\|^2 - \bar{\lambda} < Tx, x > -\lambda < x, Tx > +|\lambda|^2 \\ &\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0\end{aligned}$$

Hence $T^*x = \bar{\lambda}x$.

(2) $\lambda < x,y > = < \lambda x,y > = < Tx,y > = < x,T^*y > = < x,\bar{\mu}y > = \mu < x,y >$. Hence $< x,y > = 0$.

(3) If P is the projection of $\mathcal{H}$ onto M and $B = T|M$, then $B^* = PT^*|M$. So for x in M,

$$\|B^*x\|^2 = \|PT^*x\|^2 = \|T^*x\|^2 \leq \|T^2x\| = \|PT^2x\| = \|B^2x\|.$$

(4) Since $\lambda \in \sigma_{ap}(T)$. There exists a sequence $\{x_n\}$ of unit vector such that $\|(T - \lambda I)x_n\| \to 0$. We get $(T - \lambda I)x_n \to 0$. Since T is $*$-paranormal and $Tx_n, \lambda x_n$ are bounded sequences, we have

$$\begin{aligned}
\|T^*x_n - \bar{\lambda}x_n\|^2 &= < T^*x_n - \bar{\lambda}x_n, T^*x_n - \bar{\lambda}x_n > \\
&= \|T^*x_n\|^2 - \bar{\lambda} < x_n, T^*x_n > - \lambda < T^*x_n, x_n > + \bar{\lambda}\lambda \\
&\leq \|T^2x_n\| - \bar{\lambda} < Tx_n, x_n > - \lambda < x_n, Tx_n > + \bar{\lambda}\lambda \\
&\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0, \qquad (n \to \infty).
\end{aligned}$$

Thus $\|T^*x_n - \bar{\lambda}x_n\| \to 0, \quad (n \to \infty)$. Therefore $\bar{\lambda} \in \sigma_{ap}(T^*)$.

DEFINITION 2.2. An operator T is pure if it has no reducing subspace on which it is normal.

By simple calculations, we have following Proposition 2.3 and 2.4.

PROPOSITION 2.3. *If T is a $*$-paranormal operator, then $Ker(T-\lambda)$ reduces T.*

PROPOSITION 2.4. *If T is a $*$-paranormal operator, then $T|Ker(T-\lambda)$ is a normal operator.*

COROLLARY 2.5. *If T is a pure $*$-paranormal operator, then $\sigma_p(T) = \emptyset$.*

Proof. Suppose that $\sigma_p(T) \neq \emptyset$. By Proposition 2.3 and Proposition 2.4, we have $T|Ker(T - \lambda)$ is normal, it is a contradiction to pure. Therefore $\sigma_p(T) = \emptyset$.

LEMMA 2.6. *[7]. If $T \in L(\mathcal{H})$, the following are euivalent:*

(1) $T \in F_\ell$.
(2) $ranT$ *is closed and* $KerT$ *is finite dimensional.*
(3) *There is no sequence of unit vectors* $\{x_n\}$ *such that* $\lim \|Tx_n\| = 0$ *and* $x_n \to 0$ *weakly.*
(4) *There is no orthonormal sequence* $\{e_n\}$ *such that* $\lim \|Te_n\| \to 0$, *where* F_ℓ, F_r *and* F *denote the left Fredholm, right Fredholm, and Fredholm operators.*

LEMMA 2.7. *[7].*

(1) F_ℓ, F_r, F *are open.*
(2) $T \in F_\ell$ *if and only if* $T^* \in F_r$.

COROLLARY 2.8. *[7]. If $T \in L(\mathcal{H})$, the following are equivalent:*

(1) T *is right invertible.*
(2) T *is surjective.*
(3) $\inf\{\|T^*x\| : \|x\| = 1\} > 0$.
(4) T^* *is left invertible.*
(5) $ranT^*$ *is closed and* $Ker(T^*) = (0)$.

PROPOSITION 2.9. *Let T be a $*$-paranormal operator.*

(1) T *is invertible if and only if* T *is right invertible.*
(2) T *is Fredholm if and only if* $\pi(T)$ *has a right inverse in* $L(\mathcal{H})/K(\mathcal{H})$.
(3) $\sigma(T) = \sigma_r(T)$ *and* $\sigma_e(T) = \sigma_{re}(T)$ *(*$\sigma_r(T)$*: right spectrum).*

Proof. (1) Suppose $TB = I$ where $B \in L(\mathcal{H})$. Then $(TB)^* = B^*T^* = I$. Hence $KerT^* = (0)$; and by Proposition 2.1 (1),we have $KerT = (0)$. Hence T is injective and T is surjective by Corollary 2.8. Therefore T is invertible.

(2) By Lemma 2.6 and Lemma 2.7, $T^* \in F_\ell$ if and only if $T \in F_r$. Hence $ranT^*$ is closed and $KerT^*$ is finite dimensional, and $ranT$ is closed if and only if $ranT^*$ is closed. Since T is $*$-paranormal, $KerT$ is finite dimensional. Therefore F is Fredholm operator.

(3) This is immediate from (1) and (2).

THEOREM 2.10. *Let T be a $*$-paranormal operator. Then $\lambda \in \sigma_{ap}(T)$ if and only if there is a $*$-homomorphism $\phi : C^*(T) \to \mathbf{C}$ such that $\phi(T) = \lambda$ where $C^*(T)$ is the C^*-algebra generated by a single operator T.*

Proof. Suppose $\phi : C^*(T) \to \mathbf{C}$ is a $*$-homomorphism such that $\phi(T) = \lambda$. If $\lambda \notin \sigma_{ap}(T)$, then there is a constant $c > 0$ such that $\|(T - \lambda x\| \geq c\|x\|$ for all x in $\mathcal{H}$. This implies that $T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda - c^2$ is a positive operator. Hence $0 \leq \phi(T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda) - c^2 = -c^2$, a contradiction. Hence $\lambda \in \sigma_{ap}(T)$. Conversely, suppose $\lambda \in \sigma_{ap}(T)$. Let $\{x_n\}$ be a sequence of unit vectors in $\mathcal{H}$ such that $\|(T - \lambda)x_n\| \to 0$. Let LIM denote a Banach limit and define $\phi : L(\mathcal{H}) \to \mathbf{C}$ by $\phi(B) = LIM < Bx_n, x_n >$. If $B \in L(\mathcal{H})$, then $\|B(T - \lambda)x_n\| \to 0$. So $\phi(B(T - \lambda)) = LIM < B(T - \lambda)x_n, x_n >= 0$ Since T is $*$-paranormal, $\|(T-\lambda)^*x_n\| \to 0$. Therefore $\phi(B(T-\lambda)^*) = 0$ for every B in $L(\mathcal{H})$ and $\phi(I) = LIM\|x_n\|^2 = 1$. Therefore if $p(T, T^*)$ is any non-commuting polynomial in T and T^* that has no constant term, $\phi(p(T, T^*) + \alpha) = \alpha$ for all α in $\mathbf{C}$. This implies that ϕ is multiplicative on a dense subalgebra of $C^*(T)$. Hence $\phi|_{C^*(T)}$ is multiplicative and $\phi(T) = 0$ and

$$\begin{aligned} 0 = \phi(T - \lambda) &= LIM < (T - \lambda)x_n, x_n > \\ &= LIM < Tx_n, x_n > + LIM < -\lambda x_n, x_n > \\ &= \phi(T) - \lambda. \end{aligned}$$

So $\phi(T) = \lambda$ and $\phi(T^*) = LIM < T^*x_n, x_n >= \{LIM < Tx_n, x_n > \}^* = (\phi(T))^*$. Therefore ϕ is a $*$-homomorphism such that $\phi(T) = \lambda$.

LEMMA 2.11. *[7]. If $\mathcal{A}$ is a C^*-algebra with identity, $\Phi =$ the set of non-zero homomorphisms of $\mathcal{A}$ onto $\mathbf{C}$, and I is the commutator ideal of $\mathcal{A}$ (that is, $I =$ the norm closed ideal generated by $\{AB - BA : A, B \in \mathcal{A}\}$), then $I = \cap\{\phi^{-1}(0) : \phi \in \Phi\}$.*

PROPOSITION 2.12. *[7]. With the notation of the preceding lemma, Φ is the maximal ideal space of $\mathcal{A}/I$. Hence $\mathcal{A}/I \cong C(\Phi)$ under the Gel'fand transform, $A + I \to \widehat{A}$ where $\widehat{A}(\phi) = \phi(A)$ for A in $\mathcal{A}$ and ϕ in Φ.*

COROLLARY 2.13. *If T is $*$-paranormal, there is an isometric $*$-isomorphism $C^*(T)/I \cong C(\sigma_{ap}(T))$, where $T+I$ is sent to the function z.*

Proof. Let $\tau : \Phi \to \sigma_{ap}(T)$ be defined by $\tau(\phi) = \phi(T)$. Theorem 2.12 says that this map is surjective on the other hand, if $\phi, \psi \in \Phi$ and $\phi(T) = \psi(T)$, then $\phi = \psi$. Since Φ is compact and the map is continuous. Hence τ is a homeomorphism and then $\tau^\sharp : C(\sigma_{ap}(T)) \to C(\Phi)$ is defined by $\tau^\sharp(f) = f \circ \tau$. Note that $\tau^\sharp$ is an isometric $*$-isomorphism. We define a map $\rho : C(\sigma_{ap}(T)) \to C^*(T)/I$ so that the following diagram commutes:

$$\begin{array}{ccc} C^*(T)/I & \xrightarrow{\gamma} & C(\Phi) \\ \nwarrow \rho & & \nearrow \tau^\sharp \\ & C(\sigma_{ap}(T)) & \end{array}$$

where the Gel'fand transform $\gamma : C^*(T)/I \to C(\Phi)$ is an isometric $*$-isomorphism of $C^*(T)/I$ onto $C(\Phi)$.

DEFINITION 2.14. A operator is said to be reducible if it has a non-trivial reducing subspace. If an operator is not reducible, then it is called irreducible.

PROPOSITION 2.15. *[13]. Let $\mathcal{A}$ be a C^*-subalgebra of $L(\mathcal{H})$. If $\mathcal{A}$ is irreducible, then $\mathcal{A} = K(\mathcal{H})$.*

COROLLARY 2.16. *[13]. If $\mathcal{B}$ is an irreducible C^*-algebra of operators on $\mathcal{H}$ which contains a non-zero compact operator, then $\mathcal{B}$ contains $K(\mathcal{H})$.*

PROPOSITION 2.17. *[7]. If T is an irreducible operator such that $T^*T - TT^*$ is compact, then the commutator ideal I of $C^*(T)$ is $K(\mathcal{H})$.*

3. The images of $*$-representations of $L(\mathcal{H})$ on $\mathcal{K}$.

Let $\mathcal{H}$ be an arbitrary complex Hilbert space. In [10], we constructed an extension $\mathcal{K}$ of $\mathcal{H}$ by means of weakly convergent sequence in $\mathcal{H}$ and the Banach limit and obtained the faithful $*$-representation ϕ of $L(\mathcal{H})$ on $\mathcal{K}$.

LEMMA 3.1. *[10]. There exists a faithful $*$-representation ϕ of $L(\mathcal{H})$ on $\mathcal{K}$ with the following properties:*

(1) $\|\phi(T)\| = \|T\|$.
(2) $\sigma(T) = \sigma(\phi(T))$.
(3) $\sigma_{ap}(T) = \sigma_p(\phi(T))$.
(4) *If T is a compact operator on $\mathcal{H}$, then $\phi(T)$ is a compact operator on $\mathcal{K}$.*
(5) *If T is a Fredholm operator on $\mathcal{H}$, then $\phi(T)$ is a Fredholm operator on $\mathcal{K}$.*

PROPOSITION 3.2. *If T is a $*$-paranormal operator, then $\phi(T)$ is a $*$-paranormal operator.*

Proof. Suppose that T is a $*$-paranormal operator. Then we have $< (T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)x, x >\geq 0$ for all $x \in \mathcal{H}$, for all $x > 0$. Thus $< (T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)x_n, x_n >\geq 0$ for all $u = \{x_n\} + \mathcal{N}$, where $\mathcal{N} =$ the set of all weakly convergent sequences $\{x_n\}$ in $\mathcal{H}$ with $LIM < x_n, x_n >= 0$. Therefore

$$
\begin{aligned}
&< (\phi(T)^*\phi(T)^*\phi(T)\phi(T) - 2\lambda\phi(T)\phi(T)^* + \lambda^2 I)u, u > \\
&=< (\phi(T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)u, u > \\
&= LIM < (T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)x_n, x_n >\geq 0
\end{aligned}
$$

for all u. Hence $\phi(T)^{*2}\phi(T)^2 - 2\lambda\phi(T)\phi(T)^* + \lambda^2\phi(I) \geq 0$.

LEMMA 3.3. *[11].*

(1) *The C^*-algebra $C^*(T)$ is isometrically $*$-isomorphism to the C^*-algebra $C^*(\phi(T))$.*
(2) *If I is the maximal ideal of $C^*(T)$, then $\phi(I)$ is the maximal ideal of $C^*(\phi(T))$.*

(3) *Let $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ be the maximal ideal space of $C^*(T)$ and $C^*(\phi(T))$, respectively. Then $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ are isometrically isomorphic.*

(4) *a) $I = \cap\{f^{-1}(0) : f \in \Phi_{C^*(T)}\} \cong I' = \{h^{-1}(0) : h \in \Phi_{C^*(\phi(T))}\}$.*
b) $C^*(T)/I \cong C^*(\phi(T))/I'$.

By Lemma 3.1 and Lemma 3.3, we have the following.

COROLLARY 3.4. *If T is a $*$-paranormal operator, then $C^*(T)/I \cong C^*(\phi(T))/I' \cong C(\sigma_p(\phi(T))$.*

Clearly, we have following proposition.

PROPOSITION 3.5.

(1) *$r(\phi(T)) = r(T)$, where $r(T) =$ spectral radius of T.*
(2) *If S is normaloid in $L(\mathcal{H})$, then $\phi(S)$ is normaloid in $L(\mathcal{K})$.*
(3) *If T is convexoid in $L(\mathcal{H})$, then $\phi(T)$ is convexoid operator in $L(\mathcal{K})$.*
(4) *If T is spectraloid in $L(\mathcal{H})$, then $\phi(T)$ is spectraloid operator in $L(\mathcal{K})$.*

Proof. 1) By Lemma 3.1, we have the result.

2) If S is normaloid in $L(\mathcal{H})$, then $r(S) = \|S\|$. By (1) $r(S) = r(\phi(S))$. Hence $r(\phi(S)) = \|\phi(S)\|$.

3) Since T is convexoid, $\overline{W(T)} = \overline{W(\phi(T))}$, and $\sigma(T) = \sigma(\phi(T))$, $\overline{W(\phi(T))} = \sigma(\phi(T))$.

4) By proposition 3.5 (1).

LEMMA 3.6. *[11]. If T is an irreducible operator, then $\phi(T)$ is an irreducible operator.*

LEMMA 3.7. *[11]. If T is an irreducible operator such that $T^*T - TT^*$ is compact, then $I' = K(\mathcal{K}) \cong \phi(K(\mathcal{H})) \cong K(\mathcal{H})$, where $I' = \cap\{h^{-1}(0) : h \in \Phi_{C^*(\phi(T))}\}$.*

From the above notation, we have the following.

THEOREM 3.8. *If T is an irreducible $*$-paranormal operator such that $T^*T - TT^*$ is compact, then $\sigma_{ap}(T) = \sigma_e(T)$ and $\sigma_p(\phi(T)) = \sigma_e(\phi(T))$.*

Proof. The fact that $\sigma_{ap}(T) = \sigma_e(T)$ follows immediately from Corollary 2.13 and Proposition 2.17. The second assertion is clear from Proposition 2.17 and Lemma 3.6.

As an immediate consequence of this theorem and Lemma 3.1, we have following.

COROLLARY 3.9. *If T is an irreducible $*$-paranormal operator such that $T^*T - TT^*$ is compact, then $\sigma_e(T) = \sigma_e(\phi(T))$.*

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA