UNSOLVABLE PDO'S OF DEGENERATE TYPE

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1. Introduction

Let $A$ be a linear partial differential operator with $C^\infty$ coefficients in an open set $U$ in $\mathbb{R}^n$. Hörmander has then proved that a necessary condition for local solvability at $x_0$ is the following:

There exist constants $C, k$ and $N$ such that

$$\left| \int f v \, dx \right| \leq C \sum_{|\alpha| \leq k} \sup |D^\alpha f| \sum_{|\beta| \leq N} \sup |D^\beta A^* v|, \quad (1.1)$$

where $f, v \in C_0^\infty(U)$, $U$ is an open set containing the origin.

Here $A^*$ denotes the adjoint of $A$. Kannai[2] proved that in $\mathbb{R}^2_{t,x}, D_t + itD_x^2$ is hypoelliptic but not locally solvable on the line $x = 0$. We consider a real valued $C^\infty$ function $a(t)$ such that $a(t) = t + o(t)$ as $t \to 0$. The purpose of our paper is to show that a partial differential operator

$$A \equiv D_t + ia(t)D_x^2 \quad (1.2)$$

is still unsolvable on the line $x = 0$, even though we replace $t$ with $t +$ higher order terms. In view of (1.1) it suffices to show that for any open set $U$ containing the origin there exist functions $f_\lambda, v_\lambda$, depending on a real parameter $\lambda$ and belonging to $C_0^\infty(U)$ such that

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\[
\lim_{\lambda \to \infty} \left| \int \int f_{\lambda} v_{\lambda} \, dt \, dx \right| = \infty, \quad (1.3)
\]
\[
\limsup_{\lambda \to \infty} \sum_{k_1 + k_2 \leq k} \sup |D^{k_1}_t D^{k_2}_x f_{\lambda}| < \infty \quad \text{for every } k, \quad (1.4)
\]
\[
\limsup_{\lambda \to \infty} \sum_{l_1 + l_2 \leq l} \sup |D^{l_1}_t D^{l_2}_x A^* v_{\lambda}| < \infty \quad \text{for every } N, \quad (1.5)
\]

2. Main results

Now we state the following:

**Theorem.** The partial differential operator \(A\) is not locally solvable on the line \(x = 0\).

**Proof.** Let \(b(t) = \int_0^t a(y) \, dy\). As in [2] we choose a density function \(u_{\lambda}(t, x)\) of the following form

\[
u_{\lambda}(t, x) = \frac{1}{\sqrt{2b\lambda + 1}} \exp \left[ - \frac{2b\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b\lambda + 1)} \right]. \quad (2.1)
\]

as a solution of \(A^* u = 0\). It is enough to show that our operator is not locally solvable at the origin. Let \(U\) be an open set containing the origin and \(\delta > 0\) a fixed number such that \(\{(t, x) : \sqrt{t^2 + x^2} < \delta\} \subset U\). We may assume that \(2\delta < 1\). From conditions for \(a(t), b(t)\) it follows that there exists a constant \(c > 0\) such that

\[b(t) = ct^2 + o(t^2), \quad t \to 0.\]

Thus it is obvious that there exists a constant \(c' > 0\) such that \(b(t) \geq c't^2\) for \(t\) near 0. Without loss of generality we may assume that \(c' = 1\). Then we obtain for \(\lambda\) large enough

\[
\frac{2b\lambda^2 + x^2\lambda}{2(2b\lambda + 1)} \geq \frac{\delta^2\lambda}{2} \quad (2.2)
\]
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if $\delta \leq |t| \leq 2\delta$ or $\delta \leq |x| \leq 2\delta$. It is clear that for each $k = (k_1, k_2)$

$$D_t^{k_1} D_x^{k_2} u_\lambda(t, x)$$

$$= G_k(t, x, \lambda, \sqrt{2b\lambda + 1}) \exp \left[ \frac{-2b\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b\lambda + 1)} \right]$$

where $G_k$ is a regular function of its arguments because $2b\lambda + 1 > 0$. We consider a function in $C_0^\infty(R^2)$ such that

$$\phi(t, x) = \begin{cases} 1 & \text{if } \sqrt{t^2 + x^2} \leq 1 \\ 0 & \text{if } \sqrt{t^2 + x^2} \geq 2. \end{cases}$$

It follows that there exist constants $C_1, C_2$ depending on $l = (l_1, l_2)$ such that

$$\left| D_t^{l_1} D_x^{l_2} A^* \left[ \phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t, x) \right] \right| \leq C_1 \delta^{-(l_1 + l_2 + 2)} \lambda^{C_2} \exp \left[ \frac{-\delta^2 \lambda}{2} \right]$$

We take a function $F(t, x) \in C_0^\infty(U)$ such that

$$\int \int F(t, x) dt dx = a \neq 0.$$

It follows that

$$\lambda^2 b(\frac{t}{\lambda^2}) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus we obtain

$$\lambda^4 \lim_{\lambda \to \infty} \int \int F(\lambda^2 t, \lambda^2 x) \phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t, x) dt dx$$

$$= \int \int F(t, x) \phi(0, 0) dt dx = a.$$

For each fixed $k, N$ we take

$$f_\lambda(t, x) = \lambda^{-2k-1} F(\lambda t, \lambda x)$$

$$v_\lambda(t, x) = \lambda^{2k+6} \phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t, x).$$

Moreover it is obvious that

$$\sum_{k_1 + k_2 \leq k} \sup |D_t^{k_1} D_x^{k_2} f_\lambda(t, x)| \leq \lambda^{-1}.$$ (2.6)

From (2.2) ~ (2.6) our theorem follows.
COROLLARY. For each positive integer $m$ the partial differential operator
\[ B \equiv D_t + i(t^{2m-1} + o(t^{2m-1}))D_x^2 \]
is not locally solvable at the origin.

Proof. We sketch briefly the proof. As seen in the proof of Theorem
we choose a density function $u_\lambda(t, x)$ of the following form
\[ u_\lambda(t, x) = \frac{1}{\sqrt{2b\lambda + 1}} \exp \left[ \frac{-2b\lambda^2 - x^2 \lambda + 2ix\lambda}{2(2b\lambda + 1)} \right]. \]
as a solution of $B^*u = 0$, where $b(t) = \int_0^t a(y)\,dy$. We may assume that
$b(t) \geq t^{2m}$ for $t$ near 0.
\[ \frac{2b\lambda^2 + x^2 \lambda}{2(2b\lambda + 1)} \geq \frac{\delta^{2m} \lambda}{2} \]
if $\delta \leq |t| \leq 2\delta$ or $\delta \leq |x| \leq 2\delta$. It follows then that there exist constants
$C_1, C_2$ such that
\[ \left| D_t^{l_1} D_x^{l_2} B^* \left[ \phi \left( \frac{t}{\delta}, \frac{x}{\delta} \right) u_\lambda(t, x) \right] \right| \leq C_1 \delta^{-1(l_1+1_2+2)} \lambda C_2 \exp \left[ -\frac{\delta^{2m} \lambda}{2} \right] \]
We choose a function $F(t, x) \in C_0^\infty(U)$ such that
\[ \int \int F(t, x) \exp[-b(t) + ix] \,dt\,dx = a \neq 0. \]
Then we obtain
\[ \lim_{\lambda \to -\infty} \lambda^{1+\frac{1}{m}} \int \int F(\lambda^{\frac{1}{m}} t, \lambda x) \phi \left( \frac{t}{\delta}, \frac{x}{\delta} \right) u_\lambda(t, x) \,dt\,dx = a. \]
For each fixed $k, N$ we take
\[ f_\lambda(t, x) = \lambda^{-k-1} F(\lambda^{\frac{1}{m}} t, \lambda x) \]
\[ v_\lambda(t, x) = \lambda^{k+3-\frac{1}{m}} \phi \left( \frac{t}{\delta}, \frac{x}{\delta} \right) u_\lambda(t, x). \]
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Example.

\[ D_t + i(t + c \sum_{j=2}^{\infty} t^j)D_x^2 \]

is not locally solvable at the origin.

Remark. For an operator

\[ D_t + ia(t)b(t)D_x^2 \]

we take a density function

\[ u_{\lambda}(t, x) = \frac{1}{\sqrt{b(t)^2 \lambda + 1}} \exp \left[ \frac{-b(t)^2 \lambda^2 - x^2 \lambda + 2ix\lambda}{2(2b(t)\lambda + 1)} \right]. \]

We note that the solution of the partial differential equation

\[ A^{*}u(t, x) = f(t, x) \]
\[ u|_{t=0} = 0, \]

might be represented by its partial Fourier transform:

\[ \hat{u}(t, \xi) = i \int_{0}^{t} \exp \left[ -(b(t) - b(s))\xi^2 \right] \hat{f}(s, \xi) \, ds. \]

This formula makes sense at least for all \( f \in C_0^{\infty}(R^2). \)

References


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