Bull. Korean Math. Soc. **31** (1994), No. 2, pp. 187–192

THE STRUCTURE OF ALMOST REGULAR SEMIGROUPS

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The author extended the small properties of topological semilattices to that of regular semigroups [3]. In this paper, it could be shown that a semigroup S is almost regular if and only if $\overline{RL} = \overline{R \cap L}$ for every right ideal R and every left ideal L of S. Moreover, it has shown that the Bohr compactification of an almost regular semigroup is regular.

Throughout, a semigroup will mean a topological semigroup which is a Hausdorff space together with a continuous associative multiplication. For a semigroup S, we denote $E(S)$ by the set of all idempotents of S. An element x of a semigroup S is called *regular* if and only if $x \in xSx$. A semigroup S is termed *regular* if every element of S is regular. If $x \in S$ is regular, then there exists an element $y \in S$ such that $x = xyx$ and $y = yxy$ (y is called an *inverse* of x) If y is an inverse of x, then xy and yx are both idempotents but are not always equal. A semigroup S is termed *recurrent(or almost pointwise periodic)* at $x \in S$ if and only if for any open set U about x, there is an integer $p > 1$ such that $x^p \in U$. S is said to be *recurrent(or almost periodic)* if and only if S is recurrent at every $x \in S$. It is known that if $x \in S$ is recurrent and $\Gamma(x) = \overline{\{x, x^2, \cdots\}}$ is compact, then $\Gamma(x)$ is a subgroup of S and hence x is a regular element of S.

DEFINITION. An element x of a semigroup S is said to be *almost regular* if and only if $x \in \overline{xSx}$. And a semigroup S is said to be *almost regular* if and only if every element of S is almost regular.

EXAMPLES. (1) Regular semigroups.

(2) Let X be a locally compact Hausdorff space and denote $C(X)$ by all continuous functions from X into itself. Then $C(X)$ is a topological

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Received April 1, 1993.

This work is done under the support of TGRC–KOSEF and the Korea Research Foundation, the Ministry of Education.

semigroup under the compact–open topology and the composition multiplication. It is easy to see that $C(X)$ is an almost regular semigroup. In general, $C(X)$ is not regular. For examples, let D^n be the n–disc. Then $C(D^n)$ can not be a regular semigroup as is shown below:

In [6], it is shown that $f \in C(X)$ is regular if and only if the range of X is a retract of X and it maps some subspace of X homeomorphically onto its range. Using this result, if $n = 1$, then non–regular element of $C(D^1)$ could be found easily. Suppose $n > 1$. Consider the maps

$$D^n \xrightarrow{f} D^{n-1} \xrightarrow{g} D^{n-1}/\partial D^{n-1} \cong S^{n-1}$$

where f and g are projections, and S^{n-1} is the $n-1$ sphere. Then gf is a continuous surjection. If gf is a regular element of $C(D^n)$, then S^{n-1}, the image of gf must be a retract of $C(D^n)$. This is impossible by the Brower No Retraction Theorem. Hence $C(D^n)$ is not regular for every natural number n.

LEMMA 1. *(1) Every subsemigroup of a compact group is almost regular.*

(2) Every dense subsemigroup of a regular semigroup is almost regular.

(3) Every almost periodic semigroup is almost regular.

Proof. (1) Let G be a compact group and let S be a subsemigroup of G. Let $x \in S$. Since G is compact, $\overline{\{x, x^2, \cdots\}}$ is a compact subsemigroup with identity 1. For any open subset U containing 1, we can choose a large n such that $x^n \in U$. Thus $(Ux \cap S) \cap xSx \neq \phi$. Since U is arbitrary, $x \in \overline{xSx}$. Therefore S is almost regular.

(2) Let S be a dense subsemigroup of a regular semigroup T. Let $x \in S$ and V be an open subset of S containing x. Then we may assume that $V = U \cap S$, where U is an open set in T. Since T is regular, $x = xyx$ for some $y \in T$. By the continuity of multiplication, there exists an open set W in T such that $xWx \subset U$. Since S is dense in T, $xwx \in U \cap S = V$ for some $w \in W \cap S$. Thus S is almost regular.

(3) Let S be an almost periodic semigroup and let U be an open subset of S containing $x \in S$. Since $x \in \overline{\{x^2, x^3, \cdots\}}$ and by the continuity of multiplication, we can choose a national number $n > 1$ such that $x^n \in U$. Thus $xSx \cap U \neq \phi$ and hence S is almost regular.

LEMMA 2. *Let S, T be a semigroups and let $f : S \to T$ be a continuous homomorphism. If S is almost regular, then $f(S)$ also is.*

Proof. Let $t \in f(S)$ and U be an open subset containing t. Then there is $x \in S$ such that $x \in f^{-1}(U)$ and $f(x) = t$. Since S is almost regular and f is continuous, $xax \in f^{-1}(U)$ for some $a \in S$, and hence $t = tf(a)t \in U$. Therefore $f(S)$ is almost regular.

LEMMA 3. . *(1) If S is an almost regular subsemigroup of a semigroup T such that $\overline{S}$ is compact, then $\overline{S}$ is regular. In particular, if S is an ideal of $\overline{S}$, then S is regular.*

(2) Every compact ideal of an almost regular semigroup is regular.

Proof. (1) Let $x \in S$. Then $x \in cl_S(xSx)$ the closure of xSx in S. Since $\overline{S}$ is compact in T, $cl_S(xSx) \subset x\overline{S}x$. Thus every element of S is regular in $\overline{S}$. Now let $y \in \overline{S}$. Then there exists a net $\{y_\alpha\}$ in S such that $y_\alpha \to y$. Since y_α is regular in $\overline{S}$, $y_\alpha = y_\alpha k_\alpha y_\alpha$ for some $k_\alpha \in \overline{S}$. Since $\overline{S}$ is compact, we may assume that $k_\alpha \to k$ in $\overline{S}$. So $y = yky$ is regular. Suppose S is an ideal of $\overline{S}$. Then for $x \in S, x = xax$ for some $a \in \overline{S}$. Since $ax \in S, axa \in S$. Thus $x = x(axa)x$ and hence x is regular in S.

(2) Let I be a compact ideal of an almost regular semigroup and let $x \in I$. Since

$$xSx \subset xS(\overline{xSx}) \subset xS(\overline{ISx}) \subset xS(\overline{Ix}) = xSIx \subset xIx,$$

$x \in \overline{xSx} \subset \overline{xIx} = xIx$ and hence I is regular.

THEOREM 4. . *Let S be a locally compact almost regular semigroup such that the multiplication on S can be extended to the one-point compactication $T = S \cup \{\infty\}$. Then*

(1) $E(S) \neq \phi$.

(2) For $x \in S$, if $x\infty = \infty$ or $\infty x \infty \in S$, then x is regular in S.

Proof. (1) From Lemma 3, T is a regular semigroup. For $x \in S$, $x = xax$ for some $a \in T$. If $a \in S$, then $ax \in E(S)$. If $a = \infty$, then $\infty x \in E(T)$. Thus $\infty x = \infty$ or $\infty x \in E(S)$. If $\infty x = \infty$, then $x = x\infty x = \infty x$ and hence $x^2 = x \in E(S)$.

(2)Since $x = x\infty x$ implies that $x = x(\infty x \infty)x$, x is regular in S if $\infty x \infty \in S$.

COROLLARY. *Let S be a dense locally compact subsemigroup of a compact regular semigroup T such that the multiplication on S can be extended continuously to the one-point compactification $S \cup \{\infty\}$. If T has an identity or zero, then S is regular.*

Proof. By Lemma 1 and 3, S is almost regular and $S \cup \{\infty\}$ is regular. If T has an identity [zero], then ∞ is an identity [zero] from [1, p104]. From Theorem 4, the proof is immediate.

THEOREM 5. *Let S be a locally compact almost periodic semigroup such that the minimal ideal $M(S)$ of S is non-empty and compact. Then for each open subset V containing $M(S)$, there exists an open regular subsemigroup J such that $M(S) \subset J \subset V$.*

Proof. From [1, p129], there exists an open subset J such that $M(S) \subset J \subset V$ and J is an ideal of the compact subset $\overline{J}$ of S. Let $x \in J$. Since S is almost periodic, $U \cap \{x^2, x^3, \cdots\} \neq \phi$, for any open subset U containing x. This implies that J is an almost regular subsemigroup of S. From Lemma 3, $\overline{J}$ is regular and hence J is regular.

THEOREM 6. *The Bohr compactification of an almost regular semigroup is regular*

Proof. Let (f, B) be the Bohr compactification of an almost regular semigroup S. Then f is a continuous homomorphism from S into B and $\overline{f(S)} = B$. From Lemma 2 and 3, B is regular.

For a subset V of a semigroup S, let $V(a) = \{x : axa \in V\}$. If V is an open subset of an almost regular semigroup S containing a, then $V(a)$ is non-empty. It is clear that $V(a) \subset W(a)$, for $V \subset W$. By the continuity of the multiplication of the semigroup, the following Lemma is immediate.

LEMMA 7. *Let a be an element of an almost regular semigroup S. Then $V(a)$ is open (closed) whenever V is open (closed)*

THEOREM 8. *Let S be an almost regular semigroup and let V be an open subset of $a \in S$. If $\overline{V(a)}$ is compact, then a is regular*

Proof. Let $\mathcal{F} = \{U : U$ is open set containing a and $U \subset V\}$. Then $\overline{U(a)} \subset \overline{V(a)}$, for every $U \in \mathcal{F}$. Hence $\mathcal{F}' = \{\overline{U(a)} : U \in \mathcal{F}\}$

is a decending family of closed subset of the compact set $\overline{V(a)}$, and $\bigcap \mathcal{F}' \neq \phi$. Let $x \in \bigcap \mathcal{F}'$. Then $axa \in \overline{U}$ for all $U \in \mathcal{F}$, and hence $axa \in \bigcap\{\overline{U} : U \in \mathcal{F}\} = \{a\}$. Therefore a is a regular element of S.

K. Iseki showed that a semigroup is regular if and only if $RL = R \cap L$ for every right ideal R and every left ideal L [5] . For almost regular semigroups, the following criterion may be useful:

THEOREM 9. *A topological semigroup S is almost regular if and only if $\overline{RL} = \overline{R \cap L}$ for every right ideal R and every left ideal L of S.*

Proof. Suppose S is an almost regular semigroup. Let R and L be right and left ideal of S respectively. Since $RL \subset R \cap L, \overline{RL} \subset \overline{(R \cap L)}$. Let $x \in R \cap L$. Then $x \in \overline{xSx} \subset \overline{RSL} \subset \overline{RL}$, and hence $\overline{RL} = \overline{(R \cap L)}$. Now let $x \in S$. Since $\{x\} \cup xS$ and $\{x\} \cup Sx$ are right and left ideals of S respectively,

$$x \in \overline{xS^1 \cap S^1 x} = \overline{xS^1 S^1 x} = \overline{\{x^2\} \cup xSx} = \{x^2\} \cup \overline{xSx},$$

where $S^1 = \{1\} \cup S$. Hence $x = x^2$ or $x \in \overline{xSx}$. Therefore S is almost regular.

COROLLARY. *Let S be a connected almost regular semigroup. Then every ideal of S is connected.*

Proof. . Let J be an ideal of the connected almost regular semigroup S. Then $SJ \subset J \subset \overline{JSJ} \subset \overline{SJ}$. Let $y \in J$. Then $y^2S \subset SJS \subset SJ$ and $y^2S \cap Sx \neq \phi$ for every $x \in J$. Since $SJ = \cup\{Sx : x \in J\} \cup y^2S$ and Sx and y^2S are connected, SJ is connected. Therefore J is connected.

It is known that if x is a regular element, then the $\mathcal{D}$–class D_x containing x is regular [1], where $\mathcal{D}$ is the well–known Green's relation. This is true for the case of almost regular.

THEOREM 10. *Let x be an element of a semigroup S. Then x is almost regular if and only if D_x is almost regular.*

Proof. Let $z \in D_x$. Then $x = yx, y = xv, z = ty, y = sz$ for some $s, t, u, v \in S^1$. Then

$$z = ty = txv \in \overline{txSxv} = \overline{tyuSyuv} = \overline{zuSszuv} = \overline{zuSstyuv} = \overline{zuSstxv}$$
$$= \overline{zuSsty} = \overline{zuSsz} = \overline{zSz}.$$

Therefore z is almost regular.

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